Trefftz functions as basic functions of FEM in application to solution of inverse heat conduction problem

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Abstract

The work presents the application of heat polynomials for solving an inverse problem. The heat polynomials form the Trefftz Method for non-stationary heat conduction problem. They have been used as base functions in Finite Element Method. Application of heat polynomials permits to reduce the order of numerical integration as compared to the classical Finite Element Method with formulation of the matrix of system of equations.

1 INTRODUCTION

Heat polynomials have been presented for the first time in Reference [5], and their application is reported in References - in [1, 2, 3, 6]. These polynomials solve the heat conduction equation, and their linear combination is used to solve non-linear stationary heat conduction problems by the Trefftz method. Completeness of their polynomial basis ensures a very good approximation of the temperature function in a finite element. The essence of the work presented here consists in introducing space-time elements and in defining a functional for the solution of the non-stationary heat conduction equation by the Finite Element Method implemented on a heat polynomial basis. The resulting stiffness matrix is still symmetric and positive definite but its dimension of integration is reduced by one order.

The calculation of a boundary condition based on the measurement of the temperature in a point inside the domain is important in many applications. This inverse problem is solved here with the use of heat polynomials. An inverse problem of non-stationary heat conduction in a planar layer is resolved to investigate the numerical properties of heat polynomials.

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2 GOVERNING EQUATION AND SOLUTION OF SIMPLE PROBLEM

Consider the one-dimensional linear heat conduction equation for a layer

\[
\rho \cdot c \frac{\partial T}{\partial \tau} = \lambda \frac{\partial^2 T}{\partial x^2}, \quad \tau \in (0, \infty), \quad x \in (0, l) \tag{1}
\]

with constant coefficients \(\rho, c, \lambda\) and the following conditions:

- initial condition
  
  \[T(x, 0) = f(x), \quad x \in (0, l)\]  \(\tag{2}\)

- boundary condition (Fig. 1)
  
  \[T(x = 0, \tau) = g(\tau), \quad \tau \in (0, \infty)\]  \(\tag{3}\)
  
  \[T(x = 1, \tau) = h(\tau), \quad \tau \in (0, \infty)\]  \(\tag{4}\)

The auxiliary variables (5) are used to write the heat conduction equations in the non-dimensional form (6).

\[
\xi = \frac{x}{l}, \quad t = \frac{\lambda}{\rho \cdot c} \frac{\tau}{l^2}, \quad \xi \in (0, 1), \quad t \in (0, \infty) \tag{5}
\]

\[
\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial \xi^2}, \quad \xi \in (0, 1), \quad t \in (0, \infty) \tag{6}
\]

Let the function \(T(\xi, t) \in C^\infty\). The following result is obtained, expanding the function \(T(\xi, t)\) in a Taylor series on point \((\xi_0, t_0)\) and eliminating even derivatives on \(\xi\) using equation (6):

Figure 1:
\[ T(x, t) = T_0 \cdot 1 + \frac{\partial T}{\partial x} \bar{x} + \frac{\partial T}{\partial t} \bar{t} + \frac{\partial^2 T}{\partial x \partial t} \bar{x} \bar{t} + \frac{\partial^2 T}{\partial^2 t} \bar{t}^2 + \frac{\partial^3 T}{\partial x^2 \partial t} \bar{x}^2 \bar{t} + \frac{\partial^3 T}{\partial x \partial^2 t} \bar{x} \bar{t}^2 + \frac{\partial^3 T}{\partial^3 t} \bar{t}^3 + \frac{\partial^4 T}{\partial x^2 \partial^2 t} \bar{x}^2 \bar{t}^2 + \frac{\partial^4 T}{\partial x \partial^3 t} \bar{x} \bar{t}^3 + \frac{\partial^4 T}{\partial^4 t} \bar{t}^4 + \frac{\partial^5 T}{\partial x^3 \partial t} \bar{x}^3 \bar{t} + \frac{\partial^5 T}{\partial x^2 \partial^2 t} \bar{x}^2 \bar{t}^2 + \frac{\partial^5 T}{\partial x \partial^3 t} \bar{x} \bar{t}^3 + \frac{\partial^5 T}{\partial^5 t} \bar{t}^5 + \ldots + R_n \] (7)

\[ T_0 = T(x_0, t_0) \quad \bar{x} = x - x_0 \quad \bar{t} = t - t_0 \]

It is noted that the derivatives of equal order with regard to the same variables are located at the diagonal, from the left to the right side in (7). Hence, grouping of the terms of the expansion (7) with the same derivatives, yields the following solution for equation (6)

\[ T(x, t) = T(x_0, t_0) \cdot 1 + \frac{\partial T}{\partial x} \bar{x} + \frac{\partial T}{\partial t} \bar{t} + \frac{\partial^2 T}{\partial x \partial t} \bar{x} \bar{t} + \frac{\partial^3 T}{\partial^2 x \partial t} \bar{x}^2 \bar{t} + \frac{\partial^4 T}{\partial^3 x \partial^2 t} \bar{x}^3 \bar{t} + \frac{\partial^5 T}{\partial^4 x \partial^3 t} \bar{x}^4 \bar{t} + \frac{\partial^6 T}{\partial^5 x \partial^4 t} \bar{x}^5 \bar{t} + \frac{\partial^7 T}{\partial^6 x \partial^5 t} \bar{x}^6 \bar{t} + \frac{\partial^8 T}{\partial^7 x \partial^6 t} \bar{x}^7 \bar{t} + \frac{\partial^9 T}{\partial^8 x \partial^7 t} \bar{x}^8 \bar{t} + \frac{\partial^{10} T}{\partial^9 x \partial^8 t} \bar{x}^9 \bar{t} + \frac{\partial^{11} T}{\partial^{10} x \partial^9 t} \bar{x}^{10} \bar{t} + \frac{\partial^{12} T}{\partial^{11} x \partial^{10} t} \bar{x}^{11} \bar{t} + \ldots + S_n \] (8)

The polynomials in expansion (8) solve equation (6). Solution (8) can be written in the following form

\[ T(\xi, t) = \sum_{n=0}^{\infty} \frac{\partial^n T}{\partial \xi^k} \cdot v_n \bar{\xi}, \bar{t} = \sum_{n=0}^{\infty} \frac{\partial^{n+1} T}{\partial \xi \partial^{n+1}} \cdot v_{2n+1} \bar{\xi}, \bar{t} = \sum_{n=0}^{\infty} \frac{\partial^n T}{\partial \xi^n} \cdot v_n \bar{\xi}, \bar{t} \] (9)

where:

\[ v_n \bar{\xi}, \bar{t} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\bar{\xi}^{n-2k}}{(n-2k)!} \frac{\bar{t}^k}{k!} \] (10)

The univocal character of the Taylor expansion of function T(\xi,t) gives,

\[ T(\xi, t) = \sum_{n=0}^{\infty} \frac{\partial^n T(\xi_0, t_0)}{\partial \xi^n} \cdot v_n \bar{\xi}, \bar{t} = S_{N+1} \quad \text{and} \quad \lim_{N \to \infty} S_N = 0 \] (11)

meaning that, the system of functions \{v_0, v_1, ..., v_N, ...\} is complete.

Functions \( v_n(\bar{\xi}, \bar{t}) \) are called heat polynomials [5]. The determination of values of function \( v_n(\bar{\xi}, \bar{t}) \) using expression (10) may be charged with considerable numerical errors. The values of function \( v_n(\bar{\xi}, \bar{t}) \)
can be determined by recurrence formulae. In order to derive the recurrence formulae one should notice that the function $W = \exp(p \cdot \xi^2 + pt)$ solves equation (6) for any value of parameter $p$. The generating function, $W$, can be expanded into a power series with regard to parameter $p$:

$$ W = e^{p \xi^2 + pt} = \sum_{n=0}^{\infty} v_n (\xi, t) \cdot p^n \quad p \in R $$  \hspace{1cm} (12)

Expansion of the generating function $W$ leads to the following relations:

$$ v_0 (\xi, t) = 1, \quad v_1 (\xi, t) = \xi, \quad v_2 (\xi, t) = \frac{1}{2} \xi^2 + t, \quad v_3 (\xi, t) = \frac{1}{6} \xi^3 + t \cdot \xi $$

$$ v_4 (\xi, t) = \frac{1}{24} \xi^4 + \frac{1}{2} \xi^2 t + \frac{1}{2} t^2, \quad v_5 (\xi, t) = \frac{1}{120} \xi^5 + \frac{1}{6} \xi^3 t + \frac{1}{2} \xi t^2 $$

$$ v_6 (\xi, t) = \frac{1}{720} \xi^6 + \frac{1}{24} \xi^4 t + \frac{1}{4} \xi^2 t^2 + \frac{1}{6} t^3 $$

$$ v_7 (\xi, t) = \frac{1}{5040} \xi^7 + \frac{1}{120} \xi^5 t + \frac{1}{12} \xi^3 t^2 + \frac{1}{6} \xi t^3 $$

$$ \ldots $$

$$ v_{n+1} (\xi, t) = \frac{\xi}{n} \cdot v_n (\xi, t) + \frac{2t}{n} \cdot v_{n-1} (\xi, t), \quad n \geq 1 \hspace{1cm} (13) $$

The following relationships are satisfied for the derivatives:

$$ \frac{\partial v_n (\xi, t)}{\partial \xi} = v_{n-1} (\xi, t), \quad n \geq 1, \quad \frac{\partial v_n (\xi, t)}{\partial t} = v_{n-2} (\xi, t), \quad n \geq 2 \hspace{1cm} (14) $$

The polynomials $v_n (\xi, t)$ identically satisfy the heat conduction equation (6). The solution of the heat conduction equation can be presented in the form,

$$ T (\xi, t) = \sum_{n=0}^{\infty} \frac{\partial^n T}{\partial \xi^n} \cdot v_n \cdot \xi, \bar{t} = \sum_{n=1}^{\infty} A_n \cdot v_n \cdot \xi, \bar{t} $$  \hspace{1cm} (15)

The unknown coefficients, $A_n$, in the approximate solution,

$$ T (\xi, t) \approx \sum_{n=0}^{N-1} A_n \cdot v_n \cdot \xi, \bar{t} = \{A\}^T \cdot \{v\} $$  \hspace{1cm} (16)

are sought through the minimization of the mean square distance of solution (16) from the initial and boundary conditions. The functional form of this distance is as follows:

$$ J (\{A\}) = \int_0^1 [T (\xi, 0) - f (\xi)]^2 d\xi + \int_0^{t_1} [T (0, \tau) - g (\tau)]^2 d\tau + \int_0^{t_2} [T (1, \tau) - h (\tau)]^2 d\tau $$ \hspace{1cm} (17)

It is noted that at the initial time $t = 0$ ($t_0 = 0$):

$$ T (\xi, 0) = \sum_{n=0}^{N-1} A_n \cdot v_n (\xi - \xi_0, 0) = \sum_{n=0}^{N-1} A_n (\xi - \xi_0)^n \frac{1}{n!} = \sum_{n=0}^{N-1} A_n \frac{\xi^n}{n!} $$

Hence, the initial condition $f (\xi)$ is approximated by linear combination of functions $1, \xi, \xi^2, \xi^3, \ldots$ that leads to Hilbert matrix elements in the matrix of the coefficients $\{A\}$. Therefore, stable numerical results can be obtained for $N \approx 12$. In the case of rapid heating or cooling of bodies, the large gradients of the function $\xi$ that occur require the number $N$ of heat polynomials in approximation (16) to be increased.
This increase leads to ill-conditioning of the system of equations defining coefficients \( \{A\} \). A solution of this problem consists in dividing the solution in finite elements and approximating the solution of equation (6) in the form of linear combination of heat functions (16).

Division of the solution range in the elements is shown in Figure 2. Physical properties of heat conduction process result in the fact that both temperature and heat flux at common element boundaries are continuous functions. Due to a finite linear combination (16) of the solution approximation one may require continuity of temperature and possibly small differences in heat fluxes \( q_{j-1} - q_j \) at the common boundary. Figure 3 shows an approximation of the function \( T(\xi, t) \) in the element of 4, 6, and 8 nodes. The order of polynomial in the direction of \( t \)-axis is strictly related with the order of polynomial in the direction of \( \xi \)-axis.

Let us determine the form of the solution (16) in a finite element \( \Omega_j \) as a function of the temperature at the finite element nodes. The temperature function is expressed by the relationship,

\[
T_j(\xi, t) = \sum_{n=1}^{N} A_{n}^{j} \cdot w_n(\xi, t), \quad \xi, t \in \Omega_j, \quad \xi = \xi_j, \quad \xi_j \leq \xi \leq \xi_{j+1}
\]  

(18)
where for simplicity of programming the condition \( w_n = v_{n-1} \) has been assumed

Function \( w_n, \xi, t \) satisfies also the following equation:

\[
\frac{\partial w_n}{\partial t} = \frac{\partial^2 w_n}{\partial \xi^2}, \quad w_n = v_{n-1}, \quad n = 1, 2, \ldots
\]

The solution at the finite element nodes,

\[
T \cdot \bar{\xi}_k, t_k = T_k = \sum_{n=1}^{N} A_n \cdot w_n \cdot \bar{\xi}_k, t_k, \quad k = 1, 2, \ldots, N
\]

provides the system of equations used to define the coefficients \( \{ A_n \} \) for each element:

\[
\begin{bmatrix}
  w_1 \bar{\xi}_1, t_1 & w_2 \bar{\xi}_1, t_1 & w_3 \bar{\xi}_1, t_1 & \cdots & w_N \bar{\xi}_1, t_1 \\
  w_1 \bar{\xi}_2, t_2 & w_2 \bar{\xi}_2, t_2 & w_3 \bar{\xi}_2, t_2 & \cdots & w_N \bar{\xi}_2, t_2 \\
  w_1 \bar{\xi}_3, t_3 & w_2 \bar{\xi}_3, t_3 & w_3 \bar{\xi}_3, t_3 & \cdots & w_N \bar{\xi}_3, t_3 \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  w_1 \bar{\xi}_N, t_N & w_2 \bar{\xi}_N, t_N & w_3 \bar{\xi}_N, t_N & \cdots & w_N \bar{\xi}_N, t_N
\end{bmatrix}
\begin{bmatrix}
  A_1 \\
  A_2 \\
  A_3 \\
  \vdots \\
  A_N
\end{bmatrix} =
\begin{bmatrix}
  T^1 \\
  T^2 \\
  T^3 \\
  \vdots \\
  T^N
\end{bmatrix}
\tag{19}
\]

The solution of this equation,

\[
\{ w \} \{ A \} = \{ T \}
\]

\[
\{ A \} = [w]^{-1} \{ T \} = [U] \{ T \}, \quad A_n = \sum_{k=1}^{N} U_{nk} \cdot T^k
\]

yields the following results:

\[
T(\xi, t) = \left( \sum_{n=1}^{N} \sum_{k=1}^{N} U_{nk} \cdot T^k \right) \cdot w_n \bar{\xi}_n, t = \sum_{n=1}^{N} \sum_{k=1}^{N} U_{nk} \cdot w_n \bar{\xi}_n, t \cdot T^k = \sum_{k=1}^{N} \varphi_k \bar{\xi}_n, t \cdot T^k
\tag{20}
\]

The basic functions \( \varphi_k \bar{\xi}_n, t , i = 1, \ldots, N \) are characterized by the following properties

1. \( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial \xi^2} \varphi_k \bar{\xi}_n, t = 0 \) as \( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial \xi^2} w_n \bar{\xi}_n, t = 0 \), \( n = 1, \ldots, N \)
2. \( \varphi_k \bar{\xi}_j, t_j = \begin{cases} 1 & k = j, \quad k = 1, \ldots, N, \quad j = 1, \ldots, N \\ 0 & k \neq j \end{cases} \)
3. \( \sum_{k=1}^{N} \varphi_k \bar{\xi}_n, t = 1 \)
4. \( \frac{\partial \varphi_k \bar{\xi}_n, t}{\partial \xi} = \sum_{n=1}^{N} U_{nk} \frac{\partial w_n \bar{\xi}_n, t}{\partial \xi} = \sum_{n=1}^{N} U_{nk} \cdot w_{n-1} \bar{\xi}_n, t \)
5. \( \frac{\partial \varphi_k \bar{\xi}_n, t}{\partial t} = \sum_{n=1}^{N} U_{nk} \frac{\partial w_n \bar{\xi}_n, t}{\partial t} = \sum_{n=1}^{N} U_{nk} \cdot w_{n-2} \bar{\xi}_n, t \)
The defect of the heat flux flowing between the elements caused by the approximation process is defined in Reference [1] by:

$$
\delta \dot{q}_j(t) = -\frac{\partial T_j(\xi_j, t)}{\partial \xi} + \frac{\partial T_{j+1}(\xi_{j+1}, t)}{\partial \xi} = \dot{q}_j - \dot{q}_{j+1}
$$

(27)

The unknown temperatures at the nodes of finite element mesh are determined by minimizing the expanded functional (29) as compared to (17),

$$
J_i = \sum_{l=1}^{L} \int_{t_i}^{t_{i+k}} [T_i(\xi, t) - T_{i+1}(\xi, t)]^2 d\xi + \sum_{j=1}^{L} \int_{t_i}^{t_{i+k}} [\dot{q}_j(\tau) - \dot{q}_{j+1}(\tau)]^2 d\tau + \\
\int_{t_i}^{t_{i+k}} [T_i(1, \tau) - h(\tau)] d\tau + \sum_{j=1}^{L} \int_{t_i}^{t_{i+k}} [\dot{q}_j(\tau) - \dot{q}_{j+1}(\tau)]^2 d\tau, \quad i = 0, k, 2k, \ldots, k = N/2 - 1
$$

(29)

where $T_i(\xi, t_i)$ is the initial temperature for time $t_i$.

The minimization of functional (29) leads to a solution at each mesh node, which depends on the initial and boundary conditions assumed.

### 3 INVERSE PROBLEM

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In the case of many technological problems definition of a boundary condition is not always possible. For example, it a case of combustion chamber of a fuel engine or a rocket jet nozzle. However, temperature measurement (e.g. temperature distribution) in the proximity of the surface of known boundary condition is possible. Hence, taking into account a known initial condition, boundary condition at the surface $\xi = 0$ and temperature measurement at the surface $\xi^*$ (Fig. 1) the unknown boundary condition at the surface...
\( \xi = 1 \) is to be defined. The problem defined this way is an inverse problem, and is ill-posed in the Hadamard’s sense.

The inverse problem is solved basing on the solution of a direct problem \( T(\xi, t, h(t)) \) depending in the sought boundary condition \( h(t) \). Due to discretized presentation of the problem the function \( h(t) \) is sought for consecutive time values. Let us consider a 8-node element (Fig. 5). Then the temperatures \( h_0, h_1, h_2, h_3 \) at the surface \( \xi = 1 \) are unknown in 4 consecutive time moments \( t_i, t_{i+1}, t_{i+2}, t_{i+3}, i = 0, 3, 6, 9, \ldots \).

Let us assume that temperature values \( T^* = T(\xi^*, t^*_k) \) in the point \( \xi^* \) are known for the time \( t^*_k, k \geq 4 \). The boundary condition vector \( \{h\} \) is determined from the following functional

\[
J_E = \| T(\xi^*, t^*, \{h\}) - T^* \|^2 
\]

(30)

Minimization of the functional (30) leads to the solution of the inverse problem in the form

\[
\{T\} = [STAB] \{T_0\} + \sum_{m=1}^{M} Z_m \cdot T_m^* + \{WA\} \cdot g_0 + \{WB\} \cdot g_1 + \{WC\} \cdot g_2 + \{WD\} \cdot g_3 
\]

(31)

Figure 5: Solution in time layer \( t_i, t_{i+k} \).

Temperature disturbance \( T_m^* \) in the point \( (\xi_m, t_m) \) equal to \( \delta T_m \) results in disturbance of temperature \( T_0 \) amounting to \( \delta T_0 \) (beside the first time-layer), so, the disturbed temperature distribution may be expressed by the relationship

\[
\{T + \delta T\} = [STAB] \cdot \{T_0 + \delta T_0\} + \sum_{m=1}^{M} Z_m \cdot (T_m + \delta T_m^*) + \\
+ \{WA\} \cdot g_0 + \{WB\} \cdot g_1 + \{WC\} \cdot g_2 + \{WD\} \cdot g_3 
\]

(32)
\[
\{\delta T\} = [\text{STAB}] \cdot \{\delta T_0\} + \sum_{m=1}^{M} Z_m \cdot \delta T_m^* \tag{33}
\]

Subtracting the relationship \((31)\) from \((32)\) one gets a formula of propagation of temperature measurement error and the error of initial temperature determination onto temperature distribution in the whole area. The matrix STAB is called stability matrix. The inverse problem is stable when spectral radius \(r\) of the STAB matrix fulfills the inequality \(r < 1\).

4 NUMERICAL CALCULATIONS

For purposes of numerical calculations the temperature distribution in the points \((\xi^*, t^*)\) has been assumed as equal to the solution of a simple problem under the conditions

\[
T(\xi, 0) = T_0 = 1, \quad \frac{\partial T}{\partial \xi} \bigg|_{\xi=0} = 0, \quad \frac{\partial T}{\partial \xi} \bigg|_{\xi=1} = Bi(T - T_f), \quad Bi = \frac{\alpha \cdot l}{\lambda}
\]

where \(Bi\) is Biot’s number (3rd boundary condition) and \(T_f\) is the temperature of liquid that surrounds the layer. If \(Bi \to \infty\) the surface \(\xi = 1\) is subject to the 1st type boundary condition. The following values have been assumed for the calculations: the mesh in \(\xi\)-direction is divided into 30 parts, time step \(\Delta t = 0.5s\), the layer thickness \(l = 58mm\), thermal conductivity \(\lambda = 27 W/mK\), heat transfer coefficient \(\alpha = 20000 W/m \cdot K\), location of temperature measurement points - 2 mm from the edge \((\xi^* = 0.9655)\).

Figures 6, 7 and 8 show the pattern of heat-transfer coefficient determined on the grounds of solution of the inverse problem using the 1st type boundary condition \((4)\) for linear \((N = 4)\).

Fig 6, square \((N=6)\) Fig.7, and cubic \((N=8)\) Fig.8 approximation of \(T(\xi, t)\) function with regard to time. Stability of the method is maintained. Values of spectral radii as functions of time step are marked in Figure 9.

![Figure 6:](image-url)
5 GENERALIZATION OF THE USE OF HEAT POLYNOMIALS

Heat polynomials \( \{ \nu_n(\xi, t) \} \) may be obtained by expansion of the function \( T(\xi, t) \) into Taylor’s series, making use of the relationship between the derivatives provided by the heat equation (8). Another way for obtaining the heat polynomials consists in expansion of the generating function (12) into power series. Advantage of the first way is a possibility of determining remainder of the Taylor’s series indicating accuracy of the approximation, while, on the other hand, the generating function enables to define the relationship between consecutive functions and their derivatives, that is of considerable numerical meaning.

Expansion of \( T(\xi, \eta, t) \) into Taylor’s series enables to obtain the heat functions for two-dimensional case for which the generating function is defined by the formula [4]

\[
W(\xi, \eta, t) = e^{p\xi + p^2 t} \cdot e^{q\eta + q^2 t}, \quad p, q \in \mathbb{R},
\]

while in three-dimensional case

\[
W(\xi, \eta, \zeta, t) = e^{p\xi + p^2 t} \cdot e^{q\eta + q^2 t} \cdot e^{s\zeta + s^2 t}, \quad p, q, s \in \mathbb{R}
\]

Functions (34) and (35) serve as a basis for defining explicit form of heat polynomials in other coordinate systems. For solving two- and three-dimensional problems the error functional (29) must be generalized to multidimensional cases.

Let us consider now a dimensionless equation of heat conduction

\[
\frac{\partial T}{\partial t} = \Delta T, \quad (\xi, \eta, \zeta) \in \Omega \subset \mathbb{R}^3, \quad t > 0
\]
Figure 8:

Figure 9:
COMPARISON OF FEM METHOD
WITH CLASSICAL AND NEW BASIC FUNCTIONS

Data: differential equation \( LT = 0 \), \( L = \frac{\partial}{\partial t} + \frac{\partial^2}{\partial \xi^2} \)

with boundary-initial condition

FEM-method with classical base functions \( \psi \)

\[
\psi(\xi_i, t) = \begin{cases} 
0 & i \neq j \\
1 & i = j 
\end{cases}
\]

\( L\psi \neq 0 \)

\( T(\xi, t) = \sum_{i=1}^{N} T\psi_i(\xi, t) \)

Solution of variational equation or minimization
of variational functional
(if its form is known)

Minimization of mean square defect of heat flux
at common element boundaries

Properties of the methods

Integration over the elements
Basic functions \( \psi \) do not depend on operator \( L \) of the \( LT = 0 \) equation
Satisfaction of 1st type boundary condition
- accurate

Integration along element edges
Basic functions \( \psi \) depend on operator \( L \) of the \( LT = 0 \) equation
Satisfaction of 1st type boundary condition
- accurate
- approximate (in mean square sense)

Band structure of the stiffness matrix

Approximation of \( T(\xi, t) \) Function with regard to \( t \)
FEM: classic \( + \)
FEM: improved \( + \)

Table 1: Comparison of stability of inverse problem with using the heat polynomials for FEM and classic FEM
Table 2: Approximation order of $T(\xi, t)$ solution in a finite element with heat polynomials approximation and the classic one.

with the following conditions

- initial conditions

$$T(\xi, \eta, \zeta, 0) = T_0(\xi, \eta, \zeta)$$

- boundary conditions, Fig. 10

$$T(\xi, \eta, \zeta, t)|_{\partial \Omega_1} = f(t), \quad -\frac{\partial T}{\partial n}_{\partial \Omega_2} = q(t)$$

$$-\frac{\partial T}{\partial n}_{\partial \Omega_3} = Bi(t) \left[ T(\xi, \eta, \zeta, t)|_{\partial \Omega_3} - T_f(t) \right], \quad \partial \Omega = \partial \Omega_1 \cup \partial \Omega_2 \cup \partial \Omega_3$$

Figure 10:
The domain $\Omega$, Fig. 10, is divided into finite elements $\Omega_j$, then solution of the heat conduction equation in the element $\Omega_j$ may be presented in the following form

$$T_j(\xi, \eta, \zeta, t) = \sum_{i=1}^{N} T^i_j \cdot \varphi_{ij}(\xi, \eta, \zeta, t)$$  \hspace{1cm} (36)$$

where base functions $\varphi_{ij}$ satisfy the equation (34). Unknown temperatures $T^i_j$ in i-nodes of the $\Omega_j$ element are sought from the condition of minimization of the following functional

$$J(\{T\}) = \sum_{j=1}^{M_\Omega} \int_{\Omega_j} [T_j(\xi, \eta, \zeta, 0) - T_\infty(\xi, \eta, \zeta)]^2 d\Omega + \sum_{k=1}^{M_1} \int_{\partial \Omega_k} [T(\xi, \eta, \zeta, t) - f(t)]^2 d\partial \Omega +$$

$$+ \sum_{k=1}^{M_2} \int_{t+\Delta t}^{t} \int_{\Omega_j} \frac{\partial T}{\partial n} + q(t) \frac{\partial T}{\partial n} + \frac{\partial T}{\partial n} + Bi(t) \cdot |T(\xi, \eta, \zeta, t) - T_f(t)|^2 d\partial \Omega +$$

$$+ \sum_{j=1}^{M} \sum_{k=1}^{M_3} \int_{t+\Delta t}^{t} \int_{\Gamma_{jk}} \left[T_j(\xi, \eta, \zeta, t) - T_k(\xi, \eta, \zeta, t)\right]^2 dtd\Gamma +$$

$$+ \sum_{j=1}^{M} \sum_{k=1}^{M_3} \int_{t+\Delta t}^{t} \int_{\Gamma_{jk}} \frac{\partial T_j}{\partial n} - \frac{\partial T_k}{\partial n} \frac{\partial T}{\partial n}^2 dtd\Gamma, \quad \Omega = \bigcup_{j=1}^{M} \Omega_j. \hspace{1cm} (37)$$

Minimization of functional (37) leads to a system of equations including unknown temperatures $T^i_j$ (36) in the nodes of finite elements. The functional (37) includes integration over an element surface with regard to initial temperature, while in the case of boundary conditions and heat flux the integration is made at the element boundaries.

6 CONCLUSIONS

The introduction of functional heat flux defect at common boundaries of elements to the process of solution of the problem of heat flow gave stable solution of the inverse problem, also in the case of square and cubic approximation of the solution $T(\xi, t)$ with regard to the variable $t$.

The order of approximation of the solution $T(\xi, t)$ with regard to $\xi$ is associated with approximation order in $t$-direction.

Results of the calculations prove high effectiveness of the use of heat polynomials for purposes of solving the inverse heat conduction problems of heat flow.

References


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