9-11-97

PSC 403: HOMEWORK 1

(Due 9-18-97)

1. The seventh logical rule of replacement, contraposition, states that $p \Rightarrow q$ can be replaced by $\neg q \Rightarrow \neg p$. Use truth tables to verify that, in all four possible cases, the two statements have the same truth value. 4

2. Use the logical rules 6 from class to prove the following Theorem.

Assume

A. $\neg p \Rightarrow r$

B. $q \Rightarrow \neg r$ $\quad r \Rightarrow \neg q$

C. $\neg[\neg q \wedge r]$. $\quad q \vee \neg r$

Then p.

3. Prove DeMorgan's 18 other law for sets: $\overline{Y \cup Z} = \overline{Y} \cap \overline{Z}$.

4. Let R, P, and I represent the weak 26 preference, strict preference, and indifference of a voter. Assume

$$xIy \Leftrightarrow [xRy \wedge yRx]$$

9-11-97

and

$$xPy \Leftrightarrow [xRy \wedge \neg yRx],$$

Prove: If R is transitive (26) then P is transitive.

5. Let $f: X \to Y$ and $A, B \subseteq X$. Prove:
   1. $f(A) \setminus f(B) \subseteq f(A \setminus B)$
   2. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
   3. part ① holds for all A and B iff f is 1-1. (29)
      with equality

6. Prove: The completeness axiom (p. 43 of notes) is equivalent to: If a non-empty subset Y is bounded below, it has an infimum.

7. Let $x, y \in \mathbb{R}$. Prove:
   1. If x is rational then $x+y$ is rational iff y is rational.
   2. There is an increasing sequence $\langle x_n \rangle$ of rational numbers such that $x_n \to x$. Likewise, there is a decreasing sequence $\langle y_n \rangle$ of rational numbers such that $y_n \to x$.

9-11-97

③ If $x<y$ then there exists a rational number z such that $x<z<y$.

8. Does $\langle \frac{n}{n+1} \rangle$ converge? If not, prove it. If so, demonstrate its limit.

9. Consider $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x}{x+1} & \text{if } x \geq 0 \\ \frac{-x}{1-x} & \text{if } x < 0. \end{cases}$$

Do $\sup f(X)$, $\max f(X)$, $\inf f(X)$, $\min f(X)$ exist? If so, what are they? Is f 1-1 or onto?

10. Consider the sets $X = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$ and $Y = X \cup \{0\}$. Are these open, closed, or neither? Explain.

11. Prove the unnumbered proposition at the bottom of p.57 of the class notes. Give examples that show parts (ii) and (iii) are true only for *finite* collections $\mathcal{C}$.

9-11-97

12. Do the following problems from Binmore:

2.10 (3), (4), (5)

3.11 (2, only the first part, i.e., the induction proof)

4.6 (3)

4.20 (1)

5.21 (1)

Phoenicians symbol	(represent sound)	call it	word for	Greek shape orientation	name	Etruscans represent sound	Romans represent sound	Roman shape
K K K		ʾaleph	ox	𐤀 A A A	alpha			A A A
9 9 𐤁	b	bēth	house	𐤁 B B	beta			B B
1 ^	g	gīmel	camel	^ 1 < Γ Γ	gamma	g & k	k	< C C
△ ◁	d	dāleth	door	◁ D △ Δ	delta			D D
𐤄 𐤄	h	hē		𐤄 E E E	epsilon (simple e)			E E E
Y Y	w	wāw	hook	F	digamma (lost)		f	F F F
1 ^	g	gīmel	camel	^ 1 < Γ Γ	gamma	g & k	g	< G G
日 ⊟		hēth		日 H	(h → later ēta)			日 H H
𐤉 𐤉	y	yōdh	hand	𐤉 𐤉 𐤉 I	iōta sound (i)		i & y	I I
Y Y	w	wāw	hook	Y Y (Y)	upsilon (u)	V	upsilon	Y Y
I I Z	z	zayin		I Z	zēta	*	z	

PSC 403: Homework 1 good — great job catching up! Guo Gang

1. ✓

p	$\Rightarrow$	q
T	T	T
T	F	F
F	T	T
F	T	F

$\neg q$	$\Rightarrow$	$\neg p$
T	T	T
T	F	F
F	T	T
F	T	F

2. ✓

1. Assume $\neg p$	Assumed Premise
2. r	A, 1, Modus Ponens
3. $\neg\neg r$	2, Double negation
4. $\neg q$	B, 3, Modus Tollens
5. $\neg\neg q \vee \neg r$	C, DeMorgan's Law
6. $q \vee \neg r$	5, Double Negation
7. q	3 ~~2~~, 6, Disjunctive Syllogism
8. $\neg q \wedge q$	4, 7, Conjunction
9. p	1~8, Reductio ad Absurdum

✓

3. We have to show

① $\overline{[Y \cup Z]} \subseteq \overline{Y} \cap \overline{Z}$ and ② $\overline{Y} \cap \overline{Z} \subseteq \overline{[Y \cup Z]}$

Let's check ①. By the definition of $\subseteq$, we have to show $(\forall x \in \overline{[Y \cup Z]})\, x \in \overline{Y} \cap \overline{Z}$

So consider any $y \in \overline{[Y \cup Z]}$. By definition, $y \notin Y \cup Z = \{x \in X \mid x \in Y \vee x \in Z\}$

So $\neg[y \in Y \vee y \in Z]$ By DeMorgan's Law, $\neg[y \in Y] \wedge \neg[y \in Z]$

Using our definitions, this can be rewritten as

$$y \notin Y \quad \wedge \quad y \notin Z$$

or

$$y \in \overline{Y} \quad \wedge \quad y \in \overline{Z}$$

Finally, by definition $y \in \overline{Y} \cap \overline{Z}$

Since we assumed nothing about y except that it was an element of $\overline{[Y \cup Z]}$, we can use Universal Generalization to conclude that

$$(\forall x \in \overline{[Y \cup Z]})\, x \in \overline{Y} \cup \overline{Z},$$

which gives us ①. The proof of ② is similar

✓ close

4. If R is transitive, $(\forall x, y, z \in X)([{}_xR_y \wedge {}_yR_z] \Rightarrow {}_xR_z)$

We need to show: $(\forall x, y, z \in X)([{}_xP_y \wedge {}_yP_z] \Rightarrow {}_xP_z)$

assume: $\forall x, y, z \in X$

(please turn over)

1. $_xP_y \wedge {}_yP_z$

2. $[{}_xR_y \wedge \neg\, {}_yR_x] \wedge [{}_yR_z \wedge \neg\, {}_zR_y]$ — definition

3. $[{}_xR_y \wedge {}_yR_z] \wedge [\neg\, {}_yR_x \wedge \neg\, {}_zR_y]$ — 2, association

4. ${}_xR_z \wedge [\neg\, {}_yR_x \wedge \neg\, {}_zR_y]$ — 3, transitive

5. ${}_xR_z \wedge \neg[{}_zR_y \vee {}_yR_x]$ — DeMorgan's Law, commutation

6. ${}_xR_z \wedge \neg\, {}_zR_x$ NO — 5, transitive

7. ${}_xP_z$ — 6, definition

8. ${}_xP_y \wedge {}_yP_z \Rightarrow {}_xP_z$ — Rule of conditional Proof.

5. ① take an arbitrary $y \in f(A)$ but $y \notin f(B)$, that is $y \in f(A)\setminus f(B)$

so there exists some $x \in A$ but $x \notin B$, such that $y = f(x)$

Since $x \in A\setminus B$, $y \in f(A\setminus B)$

there fore $f(A)\setminus f(B) \subseteq f(A\setminus B)$

② take an arbitrary $x \in f^{-1}(A \cup B)$, ~~thate exists~~ some $y \in A\cup B$ such that $x = f^{-1}(y)$

If $y \in A$ then $x \in f^{-1}(A)$; if $y \in B$ then $x \in f^{-1}(B)$

Therefore $f^{-1}(A\cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

take an arbitrary $x \in f^{-1}(A) \cup f^{-1}(B)$

If $x \in f^{-1}(A)$ then there exists some $y \in A$ such that $x = f^{-1}(y)$, $y \in A$, so $y \in A\cup B$

If $x \in f^{-1}(B)$ then there exists some $y \in B$ such that $x = f^{-1}(y)$, $y \in B$, so $y \in A \cup B$

therefore $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

$f^{-1}(A\cup B) = f^{-1}(A) \cup f^{-1}(B)$

③ If f is 1-1, take an arbitrary $y \in f(A)$ but $y \notin f(B)$, that is, each $y \in f(A)\setminus f(B)$ is the image of a unique $x \in A$ but $x \notin B$ (otherwise $y \in f(B)$) such that $y = f(x)$

So each $x \in A\setminus B$ has a unique image $y \in f(A\setminus B)$, also $y \in f(A)\setminus f(B)$

therefore $f(A)\setminus f(B) \subseteq f(A\setminus B)$ holds with equality for all A and B if f is 1-1.

not quite

If f is not 1-1, take an arbitrary $y \in f(A)$ but $y \notin f(B)$, that is $y \in f(A)\setminus f(B)$, there exist two or more $x \in A$ but $x \notin B$ such that $y = f(x)$, say they are $x_1, x_2, \cdots\cdots, x_m$ $(m \geq 2)$

For each $x \in \{x_1, x_2, \cdots\cdots, x_m\ (m \geq 2)\} \subseteq A\setminus B$, there is ~~the unique~~ same image $y \in f(A\setminus B)$

There fore $f(A)\setminus f(B) \subseteq f(A\setminus B)$ does not hold with equality for all A and B if f is not 1-1.

6. Suppose a non-empty subset Y which is bounded above has a supremum, we have to show: if a non-empty subset is bounded below, it has an infinum.

Let $B = \sup Y$. Then B is the smallest number such that for any $y \in Y$, $y \leq B$

Let $X=\{x: x=-y\}$, since for any $y\in Y$, $y\leq B$, so for any $x\in X$, $x\geq -B$

Hence X is bounded below by $-B$. By the Continuum Property, X has a largest lower bound (or infimum) b. ~~we have to prove that $b=-B$~~ don't use this

~~Since $-B$ is a lower bound of X and b is the~~

Likewise we can prove that a non-empty subset Y bounded above has an supremum is equivalent to requiring a non-empty subset Y bounded below has an infimum.

7. ① We have to show two things: If x is rational, y is rational, then $x+y$ is rational; if x is rational and $x+y$ is rational, then y is rational.

(i) ~~assume that $x+y$ is not rational~~

$x=\frac{p_1}{q_1}$, $y=\frac{p_2}{q_2}$, where p_1, p_2, q_1, q_2 are all integers. $x+y=\frac{p_1q_2+p_2q_1}{q_1q_2}$

since its numerator and denominator are are intergers, $x+y$ is rational

~~this is a contradition. Hence the assumption $x+y$ is not rational is false. $x+y$ is rational~~

(ii) assume that y is not rational

$y=(x+y)-x$, where $(x+y)$ and x are all rational, so the right side of this equation is rational.

this is a contradiction. Hence the assumption y is not rational is false. y is rational.

② Construct a sequence $\langle x_n\rangle$ of rational numbers such that $x_n\geq x_{n-1}$

every $x\in\mathbb{R}$ can be written as $p+\frac{p_1}{10}+\frac{p_2}{10^2}+\frac{p_3}{10^3}+\cdots+\frac{p_n}{10^n}$, where p is an interger, and $p_1, p_2, \cdots, p_n$ are intergers such that $0\leq p_n\leq 9$ ← not necessarily finite

$\langle x_n\rangle$ can be constructed as $p,\ p+\frac{p_1}{10},\ p+\frac{p_1}{10}+\frac{p_2}{10^2},\ p+\frac{p_1}{10}+\frac{p_2}{10^2}+\frac{p_3}{10^3},\ \cdots,\ p+\frac{p_1}{10}+\frac{p_2}{10^2}+\cdots+\frac{p_m}{10^m}$

$m<n$. all the terms are rational.

Let $\varepsilon>0$ be given. $|x-x_n|=\frac{p_{m+1}}{10^{m+1}}+\frac{p_{m+2}}{10^{m+2}}+\cdots+\frac{p_n}{10^n}<\frac{1}{10^m}<\varepsilon$, $10^m>\frac{1}{\varepsilon}$

choose $N=-\ln\varepsilon$, then for any $m>N=-\ln\varepsilon$, $|x-x_n|<\frac{1}{10^m}<\varepsilon$

So there is an increasing sequence $\langle x_n\rangle$ of rational numbers such that $x_n\to x$.

Likewise we can prove that there is a decreasing sequence $\langle y_n\rangle$ of rational numbers such that $y_n\to x$.

③ Let $x=p+\frac{p_1}{10}+\frac{p_2}{10^2}+\cdots+\frac{p_n}{10^n}$, $y=q+\frac{q_1}{10}+\frac{q_2}{10^2}+\cdots+\frac{q_n}{10^n}$, $p, p_1, p_2, \cdots, p_n, q, q_1, q_2, \cdots, q_n$ are all integers, and $p_1, p_2, \cdots, p_n$ and $q_1, q_2, \cdots, q_n$ are unit integers. so $p\leq x<q\leq y$

$x<y$, so $y-x=(q-p)+\frac{q_1-p_1}{10}+\frac{q_2-p_2}{10^2}+\frac{q_3-p_3}{10^3}+\cdots+\frac{q_n-p_n}{10^n}>0$ ← how do you know you can do this?

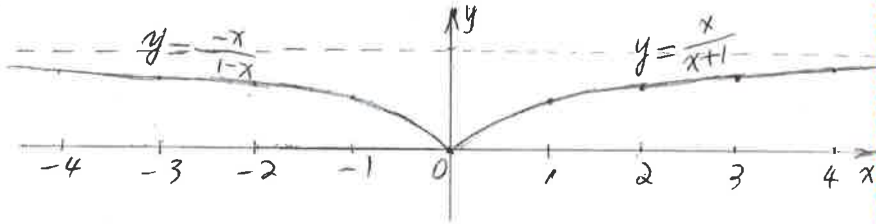
Let $t=(q-p)+\frac{q_1-p_1}{10}+\frac{q_2-p_2}{10^2}+\cdots+\frac{q_m-p_m}{10^m}$, $m<n$, so $y-x>t>0$, t is rational

that is $x<x+t<y$. when x is irrational, $x+t$ is also irrational.

Let $z=\frac{p+q}{2}+\frac{p_1+q_1}{20}+\frac{p_2+q_2}{2\times10^2}+\cdots+\frac{p_m+q_m}{2\times10^m}$, where $m<n$, z is rational; and, as $x<y$, there is at least one pair of p_m and q_m such that $p_m<\frac{p_m+q_m}{2}<q_m$

so $x<z<y$

8. $\langle \frac{n}{n+1} \rangle$ converges to 1. To see this, take any $\varepsilon > 0$, and let $N = \frac{1}{\varepsilon} - 1$, for every $n > N = \frac{1}{\varepsilon} - 1$,
$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{1}{n+1} \right| \leq \frac{1}{N+1} = \varepsilon$, as required.



9. $\sup f(x)$ exists, $\sup f(x) = 1$
$\max f(x)$ doesn't exist
$\inf f(x)$ exists, $\inf (x) = 0$
$\min f(x)$ exists, $\min (x) = 0$
f is not 1-1, take any $x_0 > 0$, $f(x_0) = \frac{x_0}{x_0+1}$; $-x_0 < 0$, $f(-x_0) = \frac{x_0}{x_0+1} = f(x_0)$.
f is not onto, take any $y_0 \geq 1$, there is no x_0 such that $f(x_0) \geq 1$

10. Consider X first.
the sequence $\langle \frac{1}{n} \rangle$ converges to 0, because, given any $\varepsilon > 0$, we can find an $N = \frac{1}{\varepsilon}$ such that, for any $n > N$, $|x_n - 0| < \varepsilon$.
As $x_n \in X$ for all n and $x_n \to 0$, but $0 \notin X$, so X is not closed. or open
Consider Y
the sequence $\langle \frac{1}{n} \rangle$ converges to 0 as above explained.
As $y_n \in Y$ for all n and $y_n \to 0$, and $0 \in Y$, so Y is closed. not open

11.(i) first consider the case $\bigcap C = \bigcap_{k=1}^{2} X_k$, that is $X_1 \cap X_2$, where X_1 and X_2 are both closed.
take a sequence $\langle x_n \rangle$ in $X_1 \cap X_2$
X_1 is closed, so $x_n \in X_1$ for all n and $x_n \to x$, then $x \in X_1$
X_2 is closed, so $x_n \in X_2$ for all n and $x_n \to x$, then $x \in X_2$
so $x \in X_1 \cap X_2$, that is, $\bigcap_{k=1}^{2} X_k$ is closed
Second, suppose $\bigcap_{k=1}^{n} X_k$ is closed, there is another closed set X_{n+1}
take a sequence $\langle x_n \rangle$ in $\bigcap_{k=1}^{n} X_k \cap X_{k+1}$
$\bigcap_{k=1}^{n} X_k$ is closed, so $x_n \in \bigcap_{k=1}^{n} X_k$ for all n and $x_n \to x$, then $x \in \bigcap_{k=1}^{n} X_k$
X_{n+1} is closed, so $x_n \in X_{k+1}$ for all n and $x_n \to x$, then $x \in X_{n+1}$
so $x \in \bigcap_{k=1}^{n} X_k \cap X_{n+1}$, that is, $\bigcap_{k=1}^{n+1} X_k$ is closed
So $\bigcap C$ is closed if C denotes an arbitrary collection of closed sets. not quite — only finite collections

(ii) first consider the case $\bigcup C = \bigcup_{k=1}^{2} X_k$, that is $X_1 \cup X_2$ where X_1 and X_2 are both closed.
take a sequence $\langle x_n \rangle$ in $X_1 \cup X_2$ x_n doesn't have to be in X_1 for all n.
X_1 is closed, so $x_n \in X_1$ for all n and $x_n \to x$, then $x \in X_1$
$x \in X_1 \subseteq X_1 \cup X_2$, that is, $\bigcup_{k=1}^{2} X_k$ is closed.
Second, suppose $\bigcup_{k=1}^{n} X_k$ is closed, there is another closed set X_{n+1}
take a sequence $\langle x_n \rangle$ in $\bigcup_{k=1}^{n} X_k \cup X_{k+1}$

$\bigcup_{k=1}^{n} X_k$ is closed, so $x_n \in \bigcup_{k=1}^{n} X_k$ for all n and $x_n \to x$, then $x \in \bigcup_{k=1}^{n} X_k$

X_{n+1} is closed, so $x_n \in X_{n+1}$ for all n and $x_n \to x$, then $x \in X_{n+1}$

$x \in \bigcup_{k=1}^{n} X_k \cup X_{n+1}$, that is, $\bigcup_{k=1}^{n+1} X_k$ is closed.

So $\bigcup C$ is closed if C denotes a finite collection of closed sets.

But: example, when $C = \{[1,2] \cup [2,3] \cup [3,4] \cup \cdots\cdots\} = [1, \infty)$ is not closed. it is closed

(iii) Let $\bigcap C = X_1 \cap X_2 \cap \cdots \cap X_n$, $X_1, X_2 \cdots X_n$ are all open sets

take an $x \in \bigcap C$, so $x \in X_1$ then there exists $\varepsilon > 0$ such that $(x-\varepsilon_1, x+\varepsilon_1) \subseteq X_1 \subseteq \bigcap C$

this is also true for $X_2, X_3, \cdots, X_n$ find smallest ε

So $\bigcap C$ is open if it denotes a finite collection of open sets.

But, when, for example, $C = \{(-0.1, 1.1) \cap (-0.01, 1.01) \cap (-0.001, 1.001) \cdots\cdots\} = [0,1]$ is closed. good

(iv) Let $\bigcup C = X_1 \cup X_2 \cup \cdots \cup X_n$, $X_1, X_2, \cdots, X_n$ are all open sets

take an $x \in X_1$, so $x \in \bigcup C$ also. then there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq X_1$

this is also true for $X_2, X_3, \cdots, X_n$

So $\bigcup C$ is open if C denotes an arbitrary collection of open sets.

Binmore:

2.10 (3)(i)(0,1)

		3 different upper bounds	the smallest upper bound	maximum
$(0,1)$	bounded above	1, 2, 3	1	does not exist
$(-\infty, 2]$	bounded above	2, 3, 4	2	2
$\{-1,0,2,5\}$	bounded above	5, 6, 7	5	5
$(3,\infty)$	unbounded above	N/A	N/A	N/A
$[0,1]$	bounded above	1, 2, 3	1	1

(4)

		3 different ~~upper~~ bounds	the largest lower bound	minimum
$(0,1)$	bounded below	0, −1, −2	0	does not exist
$(-\infty, 2]$	unbounded below	N/A	N/A	N/A
$\{-1,0,2,5\}$	bounded below	−1, −2, −3	−1	−1
$(3,\infty)$	bounded below	3, 2, 1	3	does not exist
$[0,1]$	bounded below	0, −1, −2	0	0

(5)

$\{x : x \leqslant 3 \text{ or } x = 4\}$

3.11(2) Let $P(n)$ be the statement "if $x \neq 1$, $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$"

(i) When $n=1$, $\sum_{k=0}^{1} x' = 1 + x$, $\frac{1-x^{1+1}}{1-x} = 1 + x$, hence $P(1)$ is true

(ii) Assume $P(n)$ is true,

that is, $1+x+x^2+\dots+x^n = \dfrac{1-x^{n+1}}{1-x}$

hence $\displaystyle\sum_{k=0}^{n+1} x^k = \frac{1-x^{n+1}}{1-x} + x^{n+1} = \frac{1-x^{n+1}+x^{n+1}-x^{n+1+1}}{1-x} = \frac{1-x^{(n+1)+1}}{1-x}$

So if $P(n)$ is true, then $P(n+1)$ is true

This completes the proof.

4.6(3) If $x_n \to l$ as $n\to\infty$, there exists an N such that given any $\varepsilon>0$, for any $n>N$, $|x_n - l| < \varepsilon$

not organized quite right

$|\lambda x_n - \lambda l| = |\lambda||x_n - l|$, so $|\lambda x_n - \lambda l| < \varepsilon|\lambda|$

if $\lambda > 0$, let $\varepsilon' = \varepsilon\lambda$, given any $\varepsilon' > 0$

there exists an N such that for any $n>N$, $|\lambda x_n - \lambda l| < \varepsilon'$, so $\lambda x_n \to \lambda l$ as $n \to \infty$

if $\lambda < 0$, let $\varepsilon' = -\varepsilon\lambda$, given any $\varepsilon' > 0$

there exists an N such that for any $n>N$, $|\lambda x_n - \lambda l| < \varepsilon'$, so $\lambda x_n \to \lambda l$ as $n\to\infty$

if $\lambda = 0$, $\lambda x_n = 0$, $\lambda l = 0$, $0 \to 0$, so $\lambda x_n \to \lambda l$ as $n \to \infty$

4.20(1) $\dfrac{n^3+5n^2+2}{2n^3+9} = \dfrac{1+5n^{-1}+2n^{-3}}{2+9n^{-3}} \to \dfrac{1+0+0}{2+0} = \dfrac{1}{2}$ as $n\to\infty$

From the definition of convergence, $1\to1$, $2\to2$ as $n\to\infty$; also $n^{-3}\to 0$, $n^{-1}\to 0$ as $n\to\infty$

Hence by combination theorem, $1+5n^{-1}+2n^{-3} \to 1+5\times0+2\times0 = 1$ as $n\to\infty$

Similarly, $2+9n^{-3} \to 2+9\times 0 = 2$ as $n \to\infty$

The result then follows from combination theorem (iii) $\dfrac{x_n}{y_n} \to \dfrac{l}{m}$ as $n\to\infty$ (provided that $m\neq 0$)

5.21(1) If $n>m$, $|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \dots + x_{m+1} - x_m|$

$$\begin{aligned}
&\le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + |x_{n-2} - x_{n-3}| + \dots + |x_{m+1} - x_m| \\
&\le \alpha^{n-1} + \alpha^{n-2} + \dots + \alpha^m = \alpha^m(1+\alpha+\alpha^2+\dots+\alpha^{n-m-1}) \\
&= \alpha^m \times \frac{1-\alpha^{n-m}}{1-\alpha} \le \frac{\alpha^m}{1-\alpha}
\end{aligned}$$

Let $\varepsilon > 0$ be given. Choose N so large that $\dfrac{\alpha^N}{1-\alpha} < \varepsilon$

Then, for any $n>N$ and any $m>N$

$$|x_n - x_m| \le \frac{\alpha^N}{1-\alpha} < \varepsilon$$

Thus $\langle x_n \rangle$ is a Cauchy sequence. Therefore, by theorem 5.19 in Binmore, it converges.

Example: $\langle y_n \rangle = \langle \sqrt{n} \rangle$, $\sqrt{n} \to \infty$ as $n\to\infty$ but $|\sqrt{n+1} - \sqrt{n}| \to 0$ as $n\to\infty$

9-25-97

PSC 403: HOMEWORK 2

(Due 10-2-97)

1. Prove:

   (i) If X and Y are bounded subsets of $\mathbb{R}$ then $X \cup Y$ is bounded.

   (ii) If $\mathcal{C}$ is a finite collection of compact sets then $\bigcup \mathcal{C}$ is compact.

   (iii) If $\mathcal{C}$ is an arbitrary collection of compact sets then $\bigcap \mathcal{C}$ is compact.

2. Give an example of a compact set that is not equal to a finite union of closed intervals. Note: when I say "closed interval," I include $[a,a]$, which is the same thing as $\{a\}$. So a finite set of numbers would not be an adequate example.

3. Suppose $f: \mathbb{R} \to \mathbb{R}$ is strictly increasing, and let $f^{-1}: f(\mathbb{R}) \to \mathbb{R}$ be the corresponding inverse function. Prove that f^{-1} is strictly increasing.

4. Let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions, and define the new function $h: X \to \mathbb{R}$ as follows: $h(x) = \max\{f(x), g(x)\}$. Prove that h is continuous. Use this result to prove that the "absolute value function" $|x|$ is continuous.

5. If f and g are differentiable and $h(x) = \max\{f(x), g(x)\}$ for all x, is h necessarily differentiable? Prove or supply a counterexample.

6. Let $x_n \to x$ and let $\langle y_n \rangle$ be a bounded sequence. Use the "Sandwich theorem" (Binmore's thm 4.10) to prove that,

$$(x_n - x)\, y_n \to 0.$$

7. Do the following exercises from Binmore.

8.15 (1), 8.15 (3), 9.17 (4), 10.11 (1) (you can use thm 10.9 on this one), 10.11 (5), 10.15 (5), 11.8 (1)

8. Do the following exercises from Simon & Blume.

2.11 Find the derivative of the following functions at an arbitrary point: a) $-7x^3$, b) $12x^{-2}$, c) $3x^{-3/2}$, d) $\frac{1}{2}\sqrt{x}$, e) $3x^2 - 9x + 7x^{2/5} - 3x^{1/2}$, f) $4x^5 - 3x^{1/2}$

2.11, 2.12, 2.14, 3.1

971002 Th 14:00

EXERCISES

2.10 *a*) Use the geometric definition of the derivative to prove that the derivative of a constant function is 0 everywhere and the derivative of $f(x) = mx$ is $f'(x) = m$ for all x.

b) Use the method of the proof of Theorem 2.2 to prove that the derivative of x^3 is $3x^2$ and the derivative of x^4 is $4x^3$.

2.11 Find the derivative of the following functions at an arbitrary point:

a) $-7x^3$, *b*) $12x^{-2}$,
c) $3x^{-3/2}$, *d*) $\frac{1}{2}\sqrt{x}$,
e) $3x^2 - 9x + 7x^{2/5} - 3x^{1/2}$, *f*) $4x^5 - 3x^{1/2}$,
g) $(x^2 + 1)(x^2 + 3x + 2)$, *h*) $(x^{1/2} + x^{-1/2})(4x^5 - 3\sqrt{x})$,
i) $\dfrac{x-1}{x+1}$, *j*) $\dfrac{x}{x^2+1}$,
k) $(x^5 - 3x^2)^7$, *l*) $5(x^5 - 6x^2 + 3x)^{2/3}$,
m) $(x^3 + 2x)^3(4x + 5)^2$.

2.12 Find the equation of the tangent line to the graph of the given function for the specified value of x. [Hint: Given a point on a line and the slope of the line, one can construct the equation of the line.]

a) $f(x) = x^2$, $x_0 = 3$; *b*) $f(x) = x/(x^2 + 2)$, $x_0 = 1$.

2.13 Prove parts *a* and *b* of Theorem 2.4.

2.14 In Theorem 2.3, we proved that the derivative of $y = x^k$ is $y' = kx^{k-1}$ for all positive integers k. Use the Quotient Rule, Theorem 2.4*d*, to extend this result to negative integers k. $-k>0,\ y=x^{-k},\ y'=-kx^{-k-1}$

2.5 DIFFERENTIABILITY AND CONTINUITY

As we saw in Section 2.3, a function f is differentiable at x_0 if, geometrically speaking, its graph has a tangent line at $(x_0, f(x_0))$, or analytically speaking, the limit

$$\lim_{h_n \to 0} \frac{f(x_0 + h_n) - f(x_0)}{h_n} \tag{8}$$

exists and is the same for every sequence $\{h_n\}$ which converges to 0. If a function is differentiable at every point x_0 in its domain D, we say that the function is **differentiable**. Only functions whose graphs are "smooth curves" have tangent lines everywhere; in fact, mathematicians commonly use the word "smooth" in place of the word "differentiable."

971002 Th 14:00

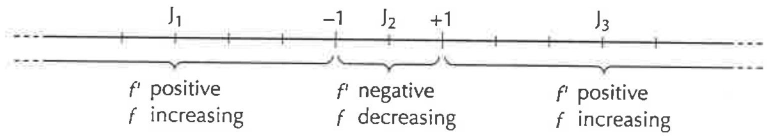


Figure 3.2 *A summary of first derivative information for* $f(x) = x^3 - 3x$.

Since it is easy to compute, you should include the y-intercept $(0, f(0))$ on the graph of f as you sketch it. The y-intercept for the function in Example 3.1 is the origin $(0, 0)$. Occasionally, it is straightforward to calculate the x-intercepts of f, the places where $f(x) = 0$. When this calculation is simple, plot these points on the graph too. For the cubic function in Example 3.1, the x-intercepts are the solutions of $f(x) = x(x^2 - 3) = 0$, namely $x = -\sqrt{3},\ 0,\ +\sqrt{3}$.

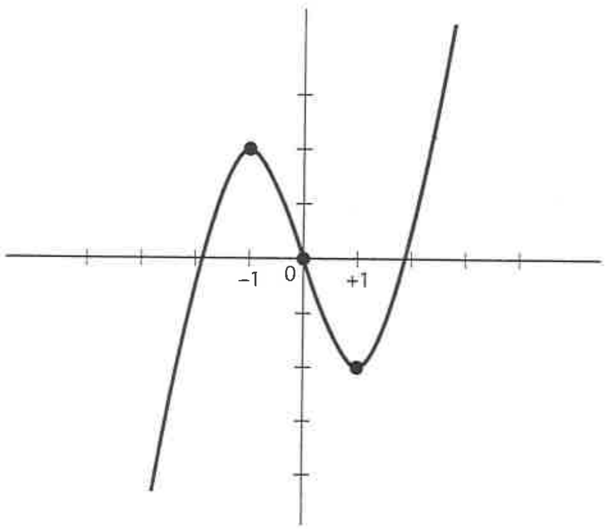


Figure 3.3 *The graph of* $f(x) = x^3 - 3x$.

EXERCISES

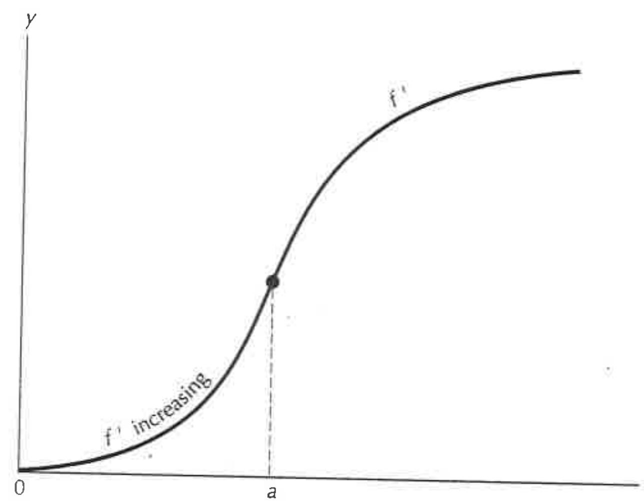
3.1 Use the techniques of this section to sketch the graphs of the following functions:

a) $x^3 + 3x$, *b)* $x^4 - 8x^3 + 18x^2 - 11$, *c)* $\frac{1}{3}x^3 + 9x + 3$,
d) $x^7 - 7x$, *e)* $x^{2/3}$, *f)* $2x^6 - 3x^4 + 2$.

3.2 Write out the corresponding argument for part *b* of Theorem 3.1.

3.2 SECOND DERIVATIVES AND CONVEXITY

Frequently, we need to know more about the shape of the graph than where it is increasing and where it is decreasing. Consider, for example, a production function $y = f(x)$, a good example of a function which is naturally increasing. The rate of increase for a production function varies with the number x of workers. At first, the additional output that each new worker adds to the production process increases as specialization and cooperation take place. However, after the gains from specialization are achieved, the additional output per new worker slows down and eventually declines as workers compete for limited space and resources. Figure 3.4 shows the graph of such a production function. Note that it is increasing for all x. However, for x between 0 and a, its slope (the marginal product of labor) is increasing too; for x bigger than a, the slope decreases as x increases.



A typical production function.

Learning curves, which relate amount learned to time elapsed, often have graphs shaped like that of Figure 3.4. Amount learned per unit time — the slope of the curve — is high at first and increasing. However, as the task becomes learned or as the learner's mind reaches its capacity to hold more data, the rate of learning begins to drop.

For $x \in (0, a)$ in Figure 3.4, the slope of $f'(x)$ is an increasing function. By Theorem 3.2, the derivative of f', $f''(x)$, is nonnegative there: $f''(x) \geq 0$ on $(0, a)$. For $x > a$ in Figure 3.4, f' is a decreasing function of x; so $f''(x) \leq 0$ on (a, ∞). A differentiable function f for which $f''(x) \geq 0$ on an interval I (so that f' is increasing on I) is said to be **concave up** on I. A differentiable function f for which $f''(x) \leq 0$ on an interval I (so that f' is decreasing on I) is said to be **concave down** on I.

An increasing function can be concave up or concave down on its interval of increase. These two cases are illustrated in Figure 3.5. Figure 3.6 shows how a

PSC 403 HMWK #2

SOLUTIONS

1. (i) Let $\bar{x}$ be an upper bound of X, $\underline{x}$ a lower bound of X, $\bar{y}$ an upper bound of Y, $\underline{y}$ a lower bound of Y. Then $\max\{\bar{x}, \bar{y}\}$ is an upper bound of $X \cup Y$, $\min\{\underline{x}, \underline{y}\}$ is a lower bound of $X \cup Y$.

(ii) Let $\mathcal{C} = \{X_1, X_2 \dots X_n\}$, where each X_i is compact, i.e., each is closed and bounded. We know that $\cup X_i$ is closed. From part (i), we know $X_1 \cup X_2$ is bounded, so is $X_1 \cup X_2 \cup X_3$, and so on. We find that $\cup X_i$ is bounded. Therefore, it is compact.

(iii) Pick any $X \in \mathcal{C}$, with upper bound $\bar{x}$ and lower bound $\underline{x}$. Then $\bar{x}$ and $\underline{x}$ are upper and lower bounds of $\cap \mathcal{C}$. Each $X \in \mathcal{C}$ is closed, so $\cap \mathcal{C}$ is closed too. Therefore, $\cap \mathcal{C}$ is compact. (NOTE: I HAD TO ASSUME $\mathcal{C}$ WAS NON-EMPTY.)

2. $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

3. Take $z, w \in f(\mathbb{R})$ with $z > w$. So there exist $x, y \in \mathbb{R}$ with $f(x) = z$ and $f(y) = w$. If $x \leq y$ then, since $x \neq y$, $x < y$. Since f is strictly increasing, $f(x) = z < w = f(y)$, a contradiction. Therefore, $f^{-1}(z) = f^{-1}(f(x)) = x > y = f^{-1}(f(y)) = f^{-1}(w)$.

4. Take $x_n \to x$. I need to show $h(x_n) \to h(x)$. Pick $\varepsilon > 0$. Since f and g are continuous, there exist $L, M \in \mathbb{N}$ such that: for all $n \geq L$, $|f(x_n) - f(x)| < \varepsilon$, and, for all $n \geq M$, $|g(x_n) - g(x)| < \varepsilon$. Define $N = \max\{L, M\}$, and take any $n \geq N$. Then $|f(x_n) - f(x)| < \varepsilon$ and $|g(x_n) - g(x)| < \varepsilon$, and $|\max\{f(x_n), g(x_n)\} - \max\{f(x), g(x)\}| < \varepsilon$. Therefore, h is continuous.

Define $f(x) = x$ and $g(x) = -x$, which are both continuous. Then $|x| = \max\{f(x), g(x)\}$ is continuous.

5. No. See $|x|$, above.

6. Let $\bar{y}$ and $\underline{y}$ be upper and lower bounds of $\langle y_n \rangle$. Note that $\langle x_n - x \rangle \to 0$, and

$$\langle x_n - x \rangle \bar{y} \to 0 \quad \text{and} \quad \langle x_n - x \rangle \underline{y} \to 0.$$

For all n, $\langle x_n - x \rangle \bar{y} \geq \langle x_n - x \rangle y_n \geq \langle x_n - x \rangle \underline{y}$. Then the sandwich theorem implies $\langle x_n - x \rangle y_n \to 0$.

Binmore problems

8.15 (1) (i) $-\frac{5}{3}$ (ii) $9/7$

8.15 (3) $f(x) = \dfrac{x^2 + 2x + 1 - 1}{x} = x + 2 \to 2$ as $x \to 0$.

9.17(4) Pick any $\hat{x} \in \mathbb{R}$ and let $\hat{y} = f(\hat{x})$. Since $f(x) \to 0$ as $x \to \infty$, there is some $\bar{x}$ such that, for all $x > \bar{x}$, $f(x) < \hat{y}$. Also, there is some $\underline{x}$ such that, for all $x < \underline{x}$, $f(x) < \hat{y}$. Clearly $\hat{x} \in [\underline{x}, \bar{x}]$. This is a compact interval. Since f is continuous, it attains a maximum on $[\underline{x}, \bar{x}]$. This is actually a maximum over all $\mathbb{R}$.

10.11 (1) $D \dfrac{1}{1+x^2} = \dfrac{(1+x^2)\cdot 0 - 1 \cdot D(1+x^2)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2}$.

10.11. (5) The First derivative at x, $P'(x)$, is

$$f'(\xi) + (x-\xi) f''(\xi) + \cdots + \frac{(x-\xi)^{n-2}}{(n-2)!} f^{n-1}(\xi).$$

When $x = \xi$, this is just $f'(\xi)$. By an induction argument (but it is obvious enough)

$$P^{k}(x) = f^{k}(\xi) + (x-\xi) f^{k+1}(\xi) + \cdots + \frac{(x-\xi)^{n-k-1}}{(n-k-1)!} f^{n-1}(\xi).$$

When $x = \xi$, $P^{k}(\xi) = f^{k}(\xi)$.

10.15 (5) Since $f(f^{-1}(y)) = y$ for all y, take the derivative of both sides, using the chain rule:

$$Df(f^{-1}(y))\, Df^{-1}(y) = 1$$

$$\Rightarrow \quad Df^{-1}(y) = \frac{1}{Df(f^{-1}(y))} = \frac{1}{Df(x)},$$

where x is $f^{-1}(y)$.

11.8(1) $f(x) = (x^2 - x)(x-2) = x^3 - x^2 - 2x^2 + 2x = x^3 - 3x^2 + 2x$

$$Df(x) = 3x^2 - 6x + 2 = 0$$

$$\Rightarrow \quad x = \frac{6 \pm \sqrt{36-24}}{6} = 1 \pm \tfrac{1}{6}\sqrt{12} = 1.58,\ .42$$

redo

$f(0) = 0$, $f(3) = 27 - 3\cdot 9 + 2\cdot 3 = 6$, $f(1.58) = 1.58^3 - 3\cdot 1.58^2 + 2\cdot 1.58$ $= 3.94 - 3\cdot 2.5 + 3.16 = -.4$, $f(.42) = .42^3 - 3\cdot .42^2 + 2\cdot .42$ $= .07 - .53 + .84 = .38$. The minimum of f on $[0,3]$ is $-.4$, attained at 1.58, and the maximum is 6.

attained at 3.

Simon + Blume problems

2.11

a) $-21x^2$

b) $-24x^{-3}$

c) $-\frac{9}{2}x^{-5/2}$

d) $\frac{1}{4}x^{-\frac{1}{2}}$

e) $6x-9+\frac{14}{5}x^{-3/5}-\frac{3}{2}x^{-\frac{1}{2}}$

f) $20x^4-\frac{3}{2}x^{-\frac{1}{2}}$

g) $2x(x^2+3x+2)+(x^2+1)(2x+3)$

h) $(\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}})(4x^5-3\sqrt{x})+(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(20x^4-\frac{3}{2}x^{-\frac{1}{2}})$

i) $\dfrac{(x+1)-(x-1)}{(x+1)^2}=\dfrac{2}{(x+1)^2}$

j) $\dfrac{(x^2+1)-2x^2}{(x^2+1)^2}$

k) $7(x^5-3x^2)^6(5x^4-6x)$

l) $\frac{10}{3}(x^5-6x^2+3x)(5x^4-12x+3)$

m) $3(x^3+2x)^2(3x+2)(4x+5)^2+(x^3+2x)^3 2(4x+5)4$

2.12

a) $Df(x)=2x$, so $Df(3)=6$. I need $6x_0+c=f(x_0)$, or $c=f(x_0)-6x_0=9-18=-9$. The equation for the tangent line is $g(x)=6x-9$.

b) $Df(x)=\dfrac{(x^2+2)-2x^2}{(x^2+2)^2}=\dfrac{2-x^2}{(x^2+2)^2}$, so $Df(1)=\dfrac{2-1}{3^2}=\dfrac{1}{9}$.

I need $\frac{1}{9}x_0+c=f(x_0)$, or $c=f(x_0)-\frac{1}{9}x_0=\frac{1}{3}-\frac{1}{9}=\frac{2}{9}$. The equation for the tangent line is $g(x)=\frac{1}{9}x+\frac{2}{9}$.

2.14 I need to show that, for $f(x) = x^k$, $Df(x) = k\,x^{k-1}$, where k is a non-negative integer. This is clearly true when $k=0$, so suppose k is negative. Then $\ell = -k$ is positive, and $f(x) = x^{-\ell} = \frac{1}{x^\ell}$. Using the quotient rule,

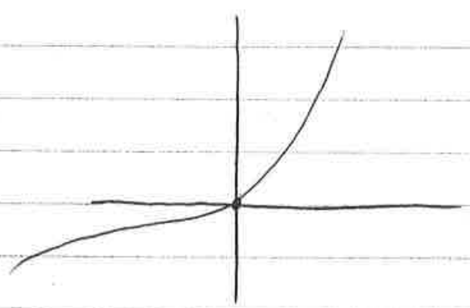
$$Df(x) = \frac{x^\ell \cdot 0 - 1 \cdot \ell x^{\ell-1}}{(x^\ell)^2} = -\ell\, x^{\ell-1-2\ell}$$

$$= -\ell\, x^{-\ell-1} = k\, x^{k-1},$$

as required.

3.1 a) $Df(x) = 3x^2 + 3 > 0$

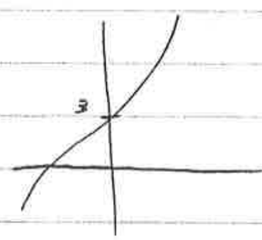
$D^2 f(x) = 6x$



b) $Df(x) = 4x^3 - 24x^2 + 18 = 0$ does not compute!

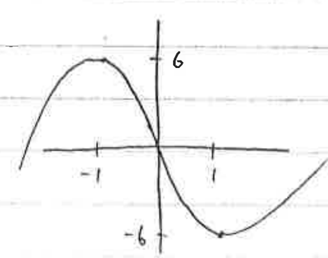
c) $Df(x) = x^2 + 9 > 0$

$D^2 f(x) = 2x$



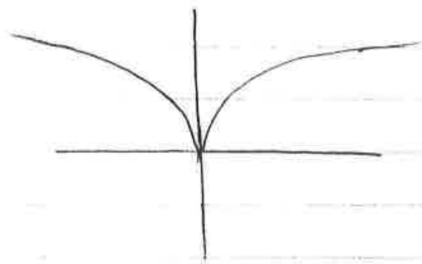
d) $Df(x) = 7x^6 - 7 = 0 \iff x \in \{1, -1\}$

$D^2 f(x) = 42x^5$



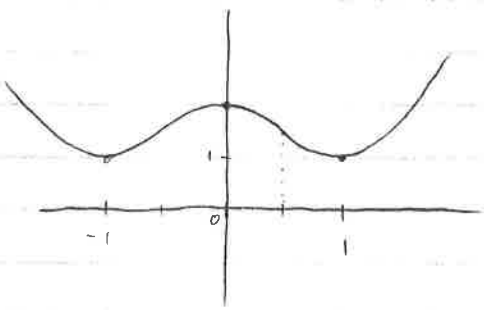
e) $Df(x) = \frac{2}{3} x^{-\frac{1}{3}}$

$D^2 f(x) = -\frac{1}{9} x^{-4/3}$



f) $Df(x) = 12x^5 - 12x^3 = 0 \iff x \in \{0, 1, -1\}$

$D^2 f(x) = 60x^4 - 36x^2 = 0 \iff x \in \{0, \pm\sqrt{\frac{36}{60}}\}$



$f(1) = 2 - 3 + 2 = 1 = f(-1)$

$f(0) = 2$

PSC 403 10-20-97

HOMEWORK # 2

(Due 10-28-97)

Optimization

1. A consumer is choosing non-negative amounts, say x and y, of two goods. A choice, denoted (x,y), is called a consumption bundle, or simply bundle. The set of bundles is $\mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the non-negative real numbers.

The consumer's preferences are represented by a utility function $u: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. That is, (x,y) is better than (x',y') iff $u(x,y) > u(x',y')$.

Let $f: u(\mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}$ be any strictly increasing function. Prove that $f \circ u$ also represents the consumer's preferences. What if f is increasing but not necessarily strictly?

2. Suppose the consumer's utility function is

$$u(x,y) = x^\alpha y^\beta,$$

where α and β are fixed, positive real numbers.

(i) Show that we can, without loss of generality, assume $0 < \alpha < 1$ and $\beta = 1 - \alpha$. That is, find a utility function $\tilde{u}(x,y) = x^{\gamma} y^{1-\gamma}$, with $0<\gamma<1$, that also represents the consumer's preferences.

The price of good x is p_x and the price of good y is p_y (both $p_x > 0$ and $p_y > 0$). The consumer has I dollars to spend. Thus, he or she must solve

$$\text{(A)} \quad \max_{(x,y)\in \mathbb{R}_+ \times \mathbb{R}_+} x^{\alpha} y^{1-\alpha}$$

subject to the constraint that

$$\text{(B)} \quad p_x x + p_y y = I.$$

(ii) Solve the "budget constraint," (B), for y and substitute this into (A) to get a "reduced" maximization problem in terms of x. Later in class, we will see that a solution to this problem necessarily exists.

(iii) Recalling that the consumer must consume non-negative amounts of the goods, what values of x is the consumer allowed to choose in the reduced problem of part (ii)?

(iv) Carefully, find a solution to the reduced problem.

You should get a formula for x in terms of prices, income, and the utility parameter α. Use your expression for y in part (ii) to get a formula for the y good. These formulas are the consumer's <u>demand functions</u> for x and y.

(v) Does the consumer's demand for x increase or decrease as p_x rises? as p_y rises? as I rises?

3. A firm produces output $q \geq 0$ at cost $C(q) = q/10$. When it produces q units, the highest price it can charge is $200 - q/10$.

(i) What is the firm's revenue, call it $R(q)$, as a function of output?

(ii) What are DC (marginal cost) and DR (marginal revenue) in terms of output? Graph these functions and indicate the firm's profit-maximizing output level.

(iii) On the same graph, indicate the firm's maximum profit level. (This will be an area.)

PSC 403 10-20-97

Integration

4. Calculate

(i) $\int_1^2 (t^5-1)\,dt$

(ii) $\int_0^4 \frac{t}{\sqrt{9+t^2}}\,dt$

(iii) $\int_0^{\sqrt{\pi}} t\cos t^2\,dt$

(iv) $\int_{-2}^3 |t|\,dt$

(v) $\int_1^4 t^2\sqrt{t^3+9}\,dt$ (integrate by substitution)

(vi) $\int_1^4 \frac{2t^3}{\sqrt{1+t^4}}\,dt$ (integrate by substitution)

(vii) $\int_1^4 t\sqrt{t+3}\,dt$ (integrate by parts)

(viii) $\int_1^4 t\ln t\,dt$ (integrate by parts)

5. The Fundamental Theorem of Calculus *seems* to say that

$$\int_{-1}^1 \frac{1}{t^2}\,dt = \left[-\frac{1}{x}\right]_{-1}^1 = -2,$$

in apparent contradiction of the fact that $\frac{1}{x^2}$ is always positive. What's going on here?

6. The density function for an exponentially distributed random variable, say $\tilde{x}$, is defined as

$$f(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where λ is some positive real number.

(i) Given a positive number a, what is the probability that the realization of $\tilde{x}$ is between 0 and a?

(ii) Confirm that $\int_{-\infty}^{\infty} f(t)\,dt = 1$.

(iii) (OPTIONAL) The expected value of $\tilde{x}$ is defined as

$$\int_{-\infty}^{\infty} t\, f(t)\, dt.$$

Check that the expected value of $\tilde{x}$ is λ. (Hints: calculate $\int_0^a t\, f(t)dt$ using integration by parts, take the limit as $a \to \infty$. using L'Hôpital's rule (see Binmore Exercise 11.8 (3)).)

PSC 403 10-20-97

Linear Algebra

7. In $\mathbb{R}^3$, draw $(1,0,0)$, $(0,1,2)$, and $(1,0,0)+(0,1,2)$.

8. Are $(1,3,3)$, $(-1,2,0)$, and $(4,-7,-1)$ linearly independent or linearly dependent?

9. Prove: If X and Y are subspaces then so is $X \cap Y$.

10. Does the following system of equations have a solution for all possible y_1, y_2, and y_3?

$$\begin{aligned} x_1 - x_2 + 4x_3 &= y_1 \\ 3x_1 + 2x_2 - 7x_3 &= y_2 \\ 3x_1 - x_3 &= y_3 \end{aligned}$$

If so, is the solution always unique?

PSC 403 Homework Oct 28,'97 GUO GANG

great

Optimization

1. $f: u(\mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}$ is strictly increasing, so iff $u(x,y) > u(x',y')$, $f\circ u(x,y) > f\circ u(x',y')$
Note that $u: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ represent consumer's preferences, iff $u(x,y) > u(x',y')$, (x,y) is better than (x',y')
So (x,y) is better than (x',y') if and only if $f\circ u(x,y) > f\circ u(x',y')$
$f\circ u$ ~~also~~ represents the consumer's preferences.
If f is not necessarily strictly increasing, $f\circ u(x,y) \geq f\circ u(x',y')$ if and only if $u(x,y) \geq u(x',y')$, (x,y) is not necessarily better than (x',y')
So $f\circ u$ does not necessarily represent the consumer's preferences.

2(i) Take $f: u(\mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R} = x^{\frac{1}{\alpha+\beta}}$, where α and β are fixed positive real numbers.
$f\circ u(x,y) = (x^\alpha y^\beta)^{\frac{1}{\alpha+\beta}} = x^{\frac{\alpha}{\alpha+\beta}} y^{1-\frac{\alpha}{\alpha+\beta}}$. Let $\gamma = \frac{\alpha}{\alpha+\beta}$, obviously $0 < \gamma < 1$
u represents consumer's preferences, and f is a strictly increasing function,
So using the result of problem 1, we know $f\circ u(x,y) = x^\gamma y^{1-\gamma}$ also represents the consumer's preferences.

(ii) $Y = \dfrac{I - P_x X}{P_y}$, Ⓐ: $\max\limits_{(x \in \mathbb{R}_+)} X^\alpha \left(\dfrac{I-P_x X}{P_y}\right)^{1-\alpha}$

(iii) $X \geq 0$ and $Y \geq 0$, so $\dfrac{I-P_x X}{P_y} \geq 0$, $X \leq \dfrac{I}{P_x}$. The consumer can choose $0 \leq X \leq \dfrac{I}{P_x}$

(iv) Solve $\max\limits_{X \in \mathbb{R}_+} X^\alpha \left(\dfrac{I-P_x X}{P_y}\right)^{1-\alpha}$. $D\left[X^\alpha\left(\dfrac{I-P_x X}{P_y}\right)^{1-\alpha}\right] = 0$, $D[X^\alpha (I-P_x X)^{1-\alpha}] = 0$

$$D(X^\alpha)(I-P_x X)^{1-\alpha} + X^\alpha D[(I-P_x X)^{1-\alpha}] = 0$$

$$\alpha X^{\alpha-1}(I-P_x X)^{1-\alpha} + X^\alpha (1-\alpha)(I-P_x X)^{-\alpha}(-P_x) = 0$$

$$\alpha(I-P_x X) - P_x X(1-\alpha) = 0, \quad X = \frac{\alpha I}{P_x}, \quad Y = \frac{I-P_x X}{P_y} = \frac{(1-\alpha)I}{P_y}$$

$$D^2\left[X^\alpha\left(\frac{I-P_x X}{P_y}\right)^{1-\alpha}\right] < 0, \quad D\left[P_y^{\alpha-1}\left(\frac{\alpha I - P_x X}{X^{\alpha-1}(I-P_x X)^{-\alpha}}\right)\right] < 0$$

(v) as P_x rises, $X = \dfrac{\alpha I}{P_x}$ decreases; as P_y rises, $Y = \dfrac{(1-\alpha)I}{P_y}$ decreases;
as I rises, $X = \dfrac{\alpha I}{P_x}$ increases.

3.(i) $R(q) = \left(200 - \frac{q}{10}\right) q = -\frac{q^2}{10} + 200q$

(ii) $DC = D\left(\frac{q}{10}\right) = \frac{1}{10}$

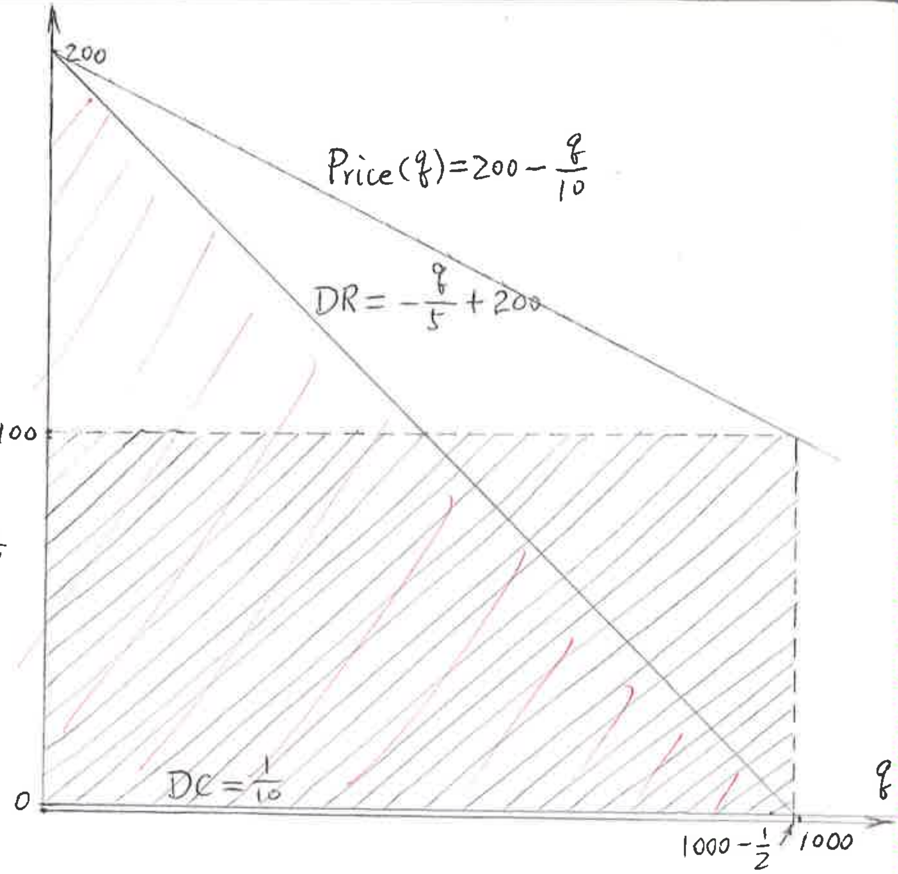
$DR = D\left(-\frac{q^2}{10} + 200q\right) = -\frac{q}{5} + 200$

$DC = DR$, $\frac{1}{10} = -\frac{q}{5} + 200$, $q = 1000 - \frac{1}{2}$

(iii) Profit $= R - C$

$= \left[-\frac{(1000-\frac{1}{2})^2}{10} + 200\left(1000 - \frac{1}{2}\right)\right] - \frac{1000 - \frac{1}{2}}{10}$

That the area of the shaded part on the graph.



Integration

4 (i) $\int_1^2 (t^5 - 1)dt$

Here $f(x) = t^5 - 1$, So $F(x) = \frac{1}{6}x^6 - x$

$\int_1^2 (t^5 - 1)dt = \left[\frac{1}{6}(2)^6 - 2\right] - \left(\frac{1}{6}\cdot 1^6 - 1\right) = \frac{57}{6}$

(ii) $\int_0^4 \frac{x}{\sqrt{9+x^2}}dx$, Let $f(x) = 9 + x^2$, $g(x) = \frac{1}{\sqrt{x}}$, of course $Df(x) = 2x$

Then $g(f(x))Df(x) = \frac{2x}{\sqrt{9+x^2}} = 2\left(\frac{x}{\sqrt{9+x^2}}\right)$

Also $f(0) = 9$, $f(4) = 25$

Then $\int_0^4 \frac{t}{\sqrt{9+t^2}}dt = \int_9^{25} \frac{1}{2}\frac{1}{\sqrt{t}}dt$, $(x) = x^{\frac{1}{2}}$

So this equals $\sqrt{25} - \sqrt{9} = 2$

(iii) Let $f(x) = x^2$, $g(x) = \cos x$, $Df(x) = 2x$

Then $g(f(x))Df(x) = 2x\cos x^2$

Also $f(0)=0$, $f(\sqrt{\pi})=\pi$

Then $\int_0^{\sqrt{\pi}} t\cos t^2\,dt = \int_0^{\pi} \frac{1}{2}\cos t\,dt = \frac{1}{2}\sin(\pi) - \frac{1}{2}\sin 0 = 0$

(iv) $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

$\int_{-2}^{3} |t|\,dt = \int_{-2}^{0} -t\,dt + \int_0^3 t\,dt = [-\frac{1}{2}t^2]_{-2}^{0} + [\frac{1}{2}t^2]_0^3 = 2 + \frac{9}{2} = \frac{13}{2}$

(v) Let $f(x)=x^3+9$, $g(x)=\sqrt{x}$, $Df(x)=3x^2$

Then $g(f(x))Df(x) = 3x^2\sqrt{x^3+9}$

Also $f(1)=10$, $f(4)=73$ 138.6

Then $\int_1^4 t^2\sqrt{t^3+9}\,dt = \int_{10}^{73} \frac{1}{3}\sqrt{t}\,dt = [\frac{2}{9}x^{\frac{3}{2}}]_{10}^{73} = \frac{146}{9}\sqrt{73} - \frac{20}{9}\sqrt{10}$

(vi) Let $f(x)=1+x^4$, $g(x)=\frac{1}{\sqrt{x}}$, $Df(x)=4x^3$

Then $g(f(x))Df(x) = \frac{4x^3}{\sqrt{1+x^4}}$

Also $f(1)=2$, $f(4)=257$

Then $\int_1^4 \frac{2t^3}{\sqrt{1+t^4}} \cancel{dt} = \int_2^{257} \frac{1}{2\sqrt{t}} \cancel{dt} = [t^{\frac{1}{2}}]_2^{257} = \sqrt{257} - \sqrt{2}$

(vii) $f(x)=x$, $G(x)=\sqrt{x+3}$, $F(x)=\frac{1}{2}x^2$, $g(x)=\frac{1}{2}(x+3)^{-\frac{1}{2}}$

$\int_1^4 t\sqrt{t+3}\,dt = (\frac{1}{2}\cdot 4^2)\sqrt{4+3} - (\frac{1}{2}\times 1^2)\sqrt{1+3} - \int_1^4 \frac{1}{4}x^2(x+3)^{-\frac{1}{2}}dx$. It's no easier. Try another way.

$f(x)=\sqrt{x+3}$, $G(x)=x$, $F(x)=\frac{2}{3}(x+3)^{\frac{3}{2}}$, $g(x)=1$

$$\int_1^4 \sqrt{t+3}\,t\,dt = \frac{2}{3}(4+3)^{\frac{3}{2}}\cdot 4 - \frac{2}{3}(1+3)^{\frac{3}{2}}\cdot 1 - \int_1^4 \frac{2}{3}(x+3)^{\frac{3}{2}}$$

$$= \frac{56}{3}\sqrt{7} - \frac{16}{3} - [\frac{4}{15}(x+3)^{\frac{5}{2}}]_1^4$$

$$= \frac{56}{3}\sqrt{7} - \frac{16}{3} - \frac{196}{15}\sqrt{7} + \frac{128}{15}$$

$$= \frac{28}{5}\sqrt{7} + \frac{16}{5}$$

(viii) $f(x)=x,\ G(x)=\ln x,\ F(x)=\frac{1}{2}x^2,\ g(x)=\frac{1}{x}$

$$\int_1^4 t\ln t\,dt=\frac{1}{2}\cdot 4^2\cdot\ln 4-\frac{1}{2}\ln 1-\int_1^4\frac{1}{2}t^2\cdot\frac{1}{t}\,dt=8\ln 4-\int_1^4\frac{t}{2}\,dt=8\ln 4-\left[\frac{t^2}{4}\right]_1^4=8\ln 4-4+\frac{1}{4}$$

5. Because $f(x)=\frac{1}{x^2}$ is not defined at $x=0$. The fundamental Theorem is not applicable here.

6.(i) We define $F(x)$ as the probability that $\tilde{x}$ takes a value less than or equal to x

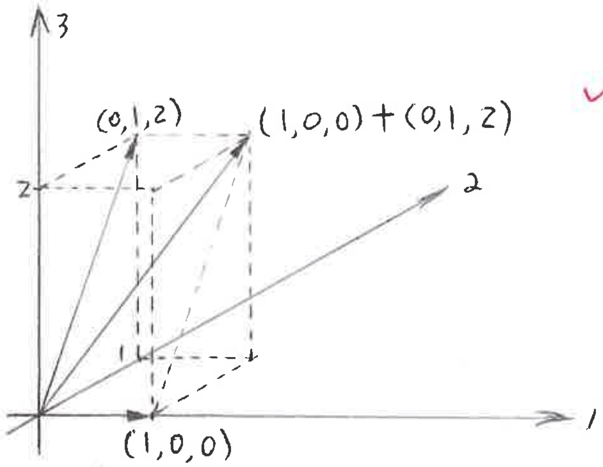
$$F(a)-F(0)=\int_0^a\frac{1}{\lambda}e^{-\frac{x}{\lambda}}\,dx=\left[-e^{-\frac{x}{\lambda}}\right]_0^a=1-e^{-\frac{a}{\lambda}}$$

(ii) $$\int_{-\infty}^{\infty}f(t)\,dt=\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}f(t)\,dt+\int_0^{\infty}f(t)\,dt=\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}0\,dt+\lim_{t\to\infty}\left[-e^{-\frac{x}{\lambda}}\right]_0^t$$

$$=-\lim_{t\to\infty}\left(\frac{1}{\sqrt[\lambda]{e^x}}\right)-\left(-e^{-\frac{0}{\lambda}}\right)=-0+1=1$$

Linear Algebra

7.



8. Suppose $\alpha_1(1,3,3)+\alpha_2(-1,2,0)+\alpha_3(4,-7,-1)=0$

$$\begin{cases}\alpha_1-\alpha_2+4\alpha_3=0 & ① \\ 3\alpha_1+2\alpha_2-7\alpha_3=0 & ② \\ 3\alpha_1-\alpha_3=0 & ③ \Rightarrow \alpha_3=3\alpha_1\end{cases}$$

$①, ③ \Rightarrow \alpha_2=13\alpha_1$; with $②, ③ \Rightarrow \alpha_1=0$, So they are linearly independent.

9. Let $X=\{\alpha_1x^1,\alpha_2x^2,\dots,\alpha_mx^m\}$, $Y=\{\beta_1y^1,\beta_2y^2,\dots,\beta_my^m\}$ $m\in\mathbb{N}$
$X\cap Y$ exists, so there is at lease one $\gamma x^k\in X$, $\gamma x^k\in Y$, $k\le m$, for all γ.
So $X\cap Y$ is span$\{x^k\}$, x^k is a basis for $X\cap Y$
$X\cap Y$ has a basis, so it is a linear subspace

see solutions

10. From the result of Problem 8, The set $\{(1,3,3),(-1,2,0),(4,-7,-1)\}$ is linearly independent, so every $y_1, y_2, y_3\in\mathrm{Span}\{(1,3,3),(-1,2,0),(4,-7,-1)\}$ is a unique linear combination of them.

PSC 403: HMW #3

SOLUTIONS

1. Suppose u represents the consumer's preferences, and let $f: u(\mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}$ be strictly increasing. Then (x,y) is better than (x',y') iff

$$u(x,y) > u(x',y')$$

$$\iff f(u(x,y)) > f(u(x',y'))$$

$$\iff (f \circ u)(x,y) > (f \circ u)(x',y'),$$

where the second line uses the fact that f is strictly increasing. Therefore, $f \circ u$ represents the consumer's preferences.

If f is increasing but not strictly, $f \circ u$ may not represent the consumer's preferences, because we can have $u(x,y) > u(x',y')$ but $f(u(x,y)) = f(u(x',y'))$.

2. (i) Define $f: \mathbb{R}_+ \to \mathbb{R}$ by $f(x) = x^{\frac{1}{\alpha+\beta}}$. This is strictly increasing, so $f \circ u$ represents the consumer's preferences. Note that

$$f(u(x,y)) = \left(x^{\alpha} y^{\beta}\right)^{\frac{1}{\alpha+\beta}} = x^{\frac{\alpha}{\alpha+\beta}} y^{\frac{\beta}{\alpha+\beta}}$$

$$= x^{\gamma} y^{1-\gamma},$$

where γ is defined as $\frac{\alpha}{\alpha+\beta}$. Setting $\tilde{u} = f \circ u$, we are done.

(ii) Solving (B),

$$y = \frac{I}{p_y} - \frac{p_x}{p_y} x.$$

Substituting, we get

$$\max \quad x^{\alpha} \left(\frac{I}{p_y} - \frac{p_x}{p_y} x \right)^{1-\alpha}.$$

(iii) Since $y \geq 0$, we know $\frac{I}{p_y} - \frac{p_x}{p_y} x \geq 0$. Solving for x,

$$x \leq \frac{I}{p_x}$$

Of course $x \geq 0$, so the consumer may choose any quantity of x in the interval $[0, I/p_x]$.

(iv) The consumer is maximizing $x^{\alpha} \left(\frac{I}{p_y} - \frac{p_x}{p_y} x \right)^{1-\alpha}$ over $[0, I/p_x]$, a differentiable function on a compact set. Therefore, there is at least one maximizer.

Can the maximizer be $x=0$ or $x = I/p_x$? No. In the first case, the consumer's utility is zero. In the second case, $y=0$ and the consumer's utility is again zero. But this is clearly not the best the consumer can do: picking $x = \frac{I}{2p_x}$, for example, yields a positive utility.

Therefore, the maximizer is an interior maximizer, so the necessary first order condition must hold:

$$\frac{d}{dx} \quad x^{\alpha} \left(\frac{I}{p_y} - \frac{p_x}{p_y} x \right)^{1-\alpha}$$

$$= \alpha x^{\alpha-1} \left(\frac{I}{p_y} - \frac{p_x}{p_y} x \right)^{1-\alpha} + x^{\alpha} (1-\alpha) \left(\frac{I}{p_y} - \frac{p_x}{p_y} x \right)^{-\alpha} \left(-\frac{p_x}{p_y} \right)$$

$$= 0.$$

Dividing by $x^{\alpha-1}\left(\frac{I}{p_y} - \frac{p_x}{p_y}x\right)$, we get

$$\alpha + \frac{x(1-\alpha)\left(-\frac{p_x}{p_y}\right)}{\left(\frac{I}{p_y} - \frac{p_x}{p_y}x\right)} = 0$$

or

$$(1-\alpha)\,x\left(\frac{p_x}{p_y}\right) = \alpha\left(\frac{I}{p_y} - \frac{p_x}{p_y}x\right)$$

or

$$(1-\alpha)\left(\frac{p_x}{p_y}\right)x = \frac{\alpha I}{p_y} - \alpha\frac{p_x}{p_y}x$$

or

$$\frac{p_x}{p_y}x = \frac{\alpha I}{p_y}$$

or

$$x = \frac{\alpha I}{p_x}.$$

Substituting into $y = \frac{I}{p_y} - \frac{p_x}{p_y}x$ and simplifying, we get

$$y = \frac{(1-\alpha)I}{p_y}.$$

(v) Demand for x decreases as p_x rises, is independent of p_y, and it increases as I rises.

3. (i) $R(q) = (200 - q/10)\,q = 200q - q^2/10$

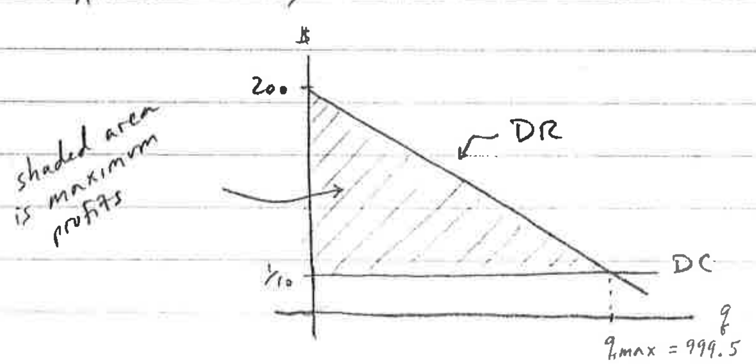
(ii),(iii) $DC = 1/10$, $DR = 200 - q/5$.

$DC = DR$

$\Rightarrow 1/10 = 200 - q/5$

$\Rightarrow q = 5(200 - 1/10)$

$= 999.5$



4. (i) $\displaystyle\int_1^2 (t^5-1)\,dt = \left[\tfrac{1}{6}x^6 - x\right]_1^2 = \left(\tfrac{1}{6}2^6 - 2\right) - \left(\tfrac{1}{6} - 1\right) = 9.5$

(ii) $\displaystyle\int_0^4 \frac{t}{\sqrt{9+t^2}}\,dt = \left[\sqrt{9+x^2}\right]_0^4 = \sqrt{9+16} - \sqrt{9} = 2$

(iii) $\displaystyle\int_0^{\sqrt{\pi}} t\cos t^2\,dt = \left[\sin x^2\right]_0^{\sqrt{\pi}} = \sin\pi - \sin 0 = 0$

(iv) $\displaystyle\int_{-2}^3 |t|\,dt = \int_{-2}^0 -t\,dt + \int_0^3 t\,dt = \left[-\tfrac{1}{2}x^2\right]_{-2}^0 + \left[\tfrac{1}{2}x^2\right]_0^3$

$$= 0 - \left(-\tfrac{1}{2}\cdot 4\right) + \tfrac{9}{2} - 0 = 6\tfrac{1}{2}$$

(v) $\displaystyle\int_1^4 t^2\sqrt{t^3+9}\,dt$ Let $f(x) = \frac{1}{3}\sqrt{x}$ and $g(x) = x^3+9$. Then $Dg(x) = 3x^2$ and

$$x^2\sqrt{x^3+9} = f(g(x))\,Dg(x).$$

Using $g(1) = 10$ and $g(4) = 73$, we have

$$\int_1^4 t^2\sqrt{t^3+9}\,dt = \int_{g(1)}^{g(4)} f(t)\,dt$$

$$= \int_{10}^{73} \tfrac{1}{3}\sqrt{t}\,dt = \tfrac{1}{3}\left[\tfrac{2}{3}x^{3/2}\right]_{10}^{73}$$

$$= \tfrac{2}{9}\left(73^{3/2} - 10^{3/2}\right) = \tfrac{2}{9}(623.71 - 31.62) = 131.57$$

(vi) $\displaystyle\int_1^4 \frac{2t^3}{\sqrt{1+t^4}}\,dt$ Let $f(x) = \frac{1}{2}x^{-\frac{1}{2}}$ and $g(x) = t^4+1$. Then $Dg(x) = 4t^3$ and

$$\frac{2x^3}{\sqrt{1+x^4}} = f(g(x))\,Dg(x).$$

Using $g(1) = 2$ and $g(4) = 257$, we have

$$\int_1^4 \frac{2t^3}{\sqrt{1+t^4}}\,dt = \int_{g(1)}^{g(4)} f(t)\,dt = \int_2^{257} \tfrac{1}{2} t^{-\frac{1}{2}}\,dt$$

$$= \tfrac{1}{2}\left[2\, t^{\frac{1}{2}}\right]_2^{257} = \sqrt{257} - \sqrt{2} = 14.62$$

(vii) $\displaystyle\int_1^4 t\sqrt{t+3}\,dt$ Let $f(x) = \sqrt{x+3}$, $g(x) = 1$, $G(x) = x$, $F(x) = \frac{2}{3}(x+3)^{3/2}$. Then

$$\int_1^4 t\sqrt{t+3}\,dt = \int_1^4 G(t)\,f(t)\,dt$$

$$= G(4)F(4) - G(1)F(1) - \int_1^4 g(t)\,F(t)\,dt$$

$$= 4\left(\tfrac{2}{3}(4+3)^{3/2}\right) - \tfrac{2}{3}(1+3)^{3/2} - \int_1^4 \tfrac{2}{3}(x+3)^{3/2}$$

$$= \tfrac{8}{3}\cdot 7^{3/2} - \tfrac{2}{3}\cdot 4^{3/2} - \tfrac{2}{3}\left[\tfrac{2}{5}(x+3)^{5/2}\right]_1^4$$

$$= 49.39 - 5.33 - \tfrac{4}{15}\left(7^{5/2} - 4^{5/2}\right)$$

$$= 44.06 - \tfrac{4}{15}(129.64 - 32)$$

$$= 18.02$$

$\frac{10}{87}\,3\sqrt{29}$ $\left(\frac{1}{29}\sqrt{29}(3x-2)\right)$
arctan $(9x^2-12x+33)$
$\frac{5}{6}\ln$

(viii) $\displaystyle\int_1^4 t\ln t\,dt$ Let $f(x) = x$, $G(x) = \ln x$, $g(x) = \frac{1}{x}$, $F(x) = \frac{1}{2}x^2$. Then

$$\int_1^4 t\ln t\,dt = \int_1^4 f(t)\,G(t)\,dt = F(4)G(4) - F(1)G(1) - \int_1^4 F(t)\,g(t)\,dt$$

$$= 8\cdot\ln 4 - 0 - \int_1^4 \tfrac{1}{2}t^2\cdot\tfrac{1}{t}\,dt = 8\ln 4 - \int_1^4 \tfrac{1}{2}t\,dt$$

$$= 8\ln 4 - \left[\tfrac{1}{4}t^2\right]_1^4 = 8\ln 4 - \left(4 - \tfrac{1}{4}\right) = 11.09 - 3.75$$

$$= 7.34.$$

5. The Fundamental Theorem of Calculus only applies if $f:[a,b] \to \mathbb{R}$ is continuous. But the function in the example isn't even defined when $x=0$. So the F.T.C. doesn't apply.

6. (i) The probability is

$$\int_0^a f(t)\,dt = \int_0^a \frac{1}{\lambda} e^{-t/\lambda}\,dt = \left[-e^{-\frac{x}{\lambda}}\right]_0^a$$

$$= -e^{-a/\lambda} + 1.$$

(ii) Note that

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_0^{\infty} f(t)\,dt$$

$$= \int_0^{\infty} f(t)\,dt = \lim_{a\to\infty} \left(-e^{-a/\lambda}+1\right).$$

Since $\lim_{a\to\infty} e^{-a/\lambda} = 0$, density integrates to one.

(iii)
$$\int_0^a t\,f(t)\,dt = \int_0^a \frac{1}{\lambda} t\,e^{-t/\lambda}\,dt.$$

Let $\hat{f}(x) = \frac{1}{\lambda}e^{-t/\lambda}$, $G(x)=x$, $g(x)=1$, $\hat{F}(x) = -e^{-t/\lambda}$.
Integrating by parts,

$$\int_0^a \frac{1}{\lambda} t\,e^{-t/\lambda}\,dt = \hat{F}(a)G(a) - \hat{F}(0)G(0) - \int_0^a \hat{F}(t)g(t)\,dt$$

$$= -e^{-a/\lambda}a - \int_0^a -e^{-t/\lambda}\,dt$$

$$= -e^{-a/\lambda}a + \int_0^a e^{-t/\lambda}\,dt$$

$$= -e^{-a/\lambda}a + \left[-\lambda e^{-t/\lambda}\right]_0^a$$

$$= -e^{-a/\lambda}a + -\lambda e^{-a/\lambda} + \lambda.$$

Now taking the limit as $a \to \infty$, note that $\lim -\lambda e^{-a/\lambda} = 0$. The limit of $a/e^{a/\lambda}$ appears indeterminate. But, using L'Hopital's rule,

$$\lim_{a\to\infty} \frac{a}{e^{a/\lambda}} = \lim_{a\to\infty} \frac{1}{\frac{1}{\lambda}e^{a/\lambda}} = 0.$$

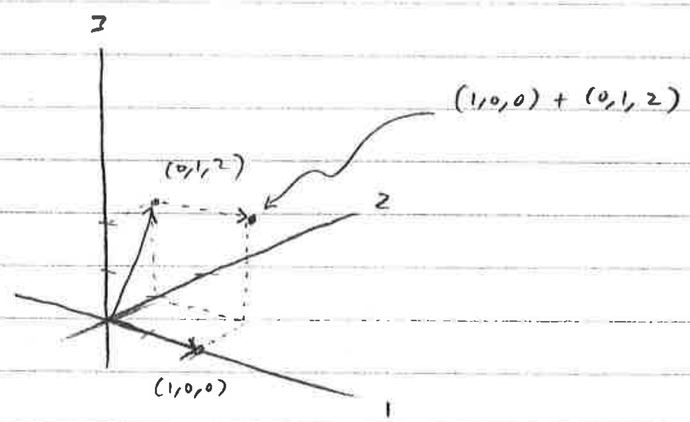
Therefore,

$$\int_{-\infty}^{\infty} t f(t)\,dt = \int_{-\infty}^{0} t f(t)\,dt + \int_{0}^{\infty} t f(t)\,dt$$

$$= 0 + \lim_{a\to\infty}\left(-e^{-a/\lambda}a - \lambda e^{-a/\lambda} + \lambda\right)$$

$$= \lambda.$$

7.



8. Check:

$$a(1,3,3) + b(-1,2,0) + c(4,-7,-1) = 0$$

$\Rightarrow$ $a - b + 4c = 0$ ①

$3a + 2b - 7c = 0$ ②

$3a - c = 0$ ③

$\Rightarrow$ $c = 3a$ ④ From ③

$\Rightarrow$ $a - b + 12a = 13a - b = 0$ ⑤ From ④ and ①

$\Rightarrow$ $b = 13a$ ⑥ From ⑤

$\Rightarrow$ $3a + 26a - 9a = 0$ ⑦ From ⑥, ④, ②

$\Rightarrow$ $a = 0$ ⑧ From ⑦

$\Rightarrow$ $b = c = 0$ ⑨ From ⑦, ⑥, ④.

The vectors are independent.

9. Take any $\alpha_1, \cdots \alpha_m \in \mathbb{R}$ and any $x^1, \cdots x^m \in X \cap Y$. Since X is a subspace, $\alpha_1 x^1 + \cdots + \alpha_m x^m \in X$. Since Y is a subspace, $\alpha_1 x^1 + \cdots + \alpha_m x^m \in Y$. Therefore

$$\alpha_1 x^1 + \cdots + \alpha_m x^m \in X \cap Y,$$

and $X \cap Y$ is a subspace.

10. The system can be rewritten as

$$\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} x_1 + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix} x_3 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

In problem # 8, we saw the column vectors were linearly independent. Therefore, every $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ is a unique linear combination of the columns. A solution always exists and is unique.

PSC 403 11-4-97

HOMEWORK #4

(Due 11-13-97)

1. Calculate the following.
   (a) $\|(3,-4,5,-1)\|$
   (b) $\|(6,2,-3)+2(-1,-1,4)\|$
   (c) the distance between $(6,2,-3)$ and $(2,-4,3)$.
   (d) $\langle(6,2,-3),(4,-6,7)\rangle$

2. Show by example that, for $x \neq 0$, there may exist more than one vector y such that $\langle x,y\rangle = 1$.

3. Call $x^1, x^2, \ldots, x^m$ <u>mutually orthogonal</u> if, for all x^i and all $x^j \neq x^i$, $\langle x^i, x^j\rangle = 0$. Prove that if $x^1, x^2, \ldots x^m$ are mutually orthogonal — and none of them equal to zero — then $\{x^1, x^2, \ldots x^m\}$ is linearly independent.

4. Given $x, y \in \mathbb{R}^n$, show that x and $y - \frac{\langle x,y\rangle}{\langle x,x\rangle} x$ are orthogonal. What is the geometric interpretation of the vector $\frac{\langle x,y\rangle}{\langle x,x\rangle} x$?

5. Consider the following system of linear equations.

$$4x_1 + 8x_2 = -20$$
$$-5x_1 - 10x_2 = 25$$

Graph the solutions to the corresponding homogeneous system of equations, and graph the solutions to the original system. Finally, plot the "row vectors" (4,8) and (-5,-10). What can you say about their relationship to the solutions of the homogeneous system?

6. Simon + Blume, Exercise 8.1. Note that the transpose of a matrix A, denoted A^T, is just the result of interchanging the rows and columns of A. For example,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

If A is $m \times n$ then A^T will be $n \times m$.

7. Letting A be an arbitrary 3×3 matrix and letting $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, check that AE is just the result of interchanging the first two columns of A. What is EA?

8. Simon + Blume, Exercises 8.15 (p.172), 8.28 (p.173), and, if we get this far in class, 9.13 (p.196).

PSC403 Hmwk #4 GUO GANG

good

1.(a) $\|(3,-4,5,-1)\| = \sqrt{3^2+(-4)^2+5^2+(-1)^2} = \sqrt{51}$

(b) $\|(6,2,-3)+2(-1,-1,4)\| = \|(4,0,5)\| = \sqrt{4^2+0^2+5^2} = \sqrt{41}$

(c) $\|(6,2,-3)-(2,-4,3)\| = \|(4,6,-6)\| = \sqrt{4^2+6^2+(-6)^2} = \sqrt{88} = 2\sqrt{22}$

(d) $\langle (6,2,-3),(4,-6,7)\rangle = 6\times 4 + 2\times(-6) + (-3)\times 7 = -9$

2. Take $(x,0,0)$ in R^3. Take $y = (\frac{1}{x}, y_2, y_3)$

$\langle x, y\rangle = x \times \frac{1}{x} + 0 y_2 + 0 \cdot y_3 = 1$, y_2, y_3 can take any value.

So y can be more than one.

Geometrically, y can be any point in the plane $x_1 = \frac{1}{x}$ which is perpendicular to axis x_1.

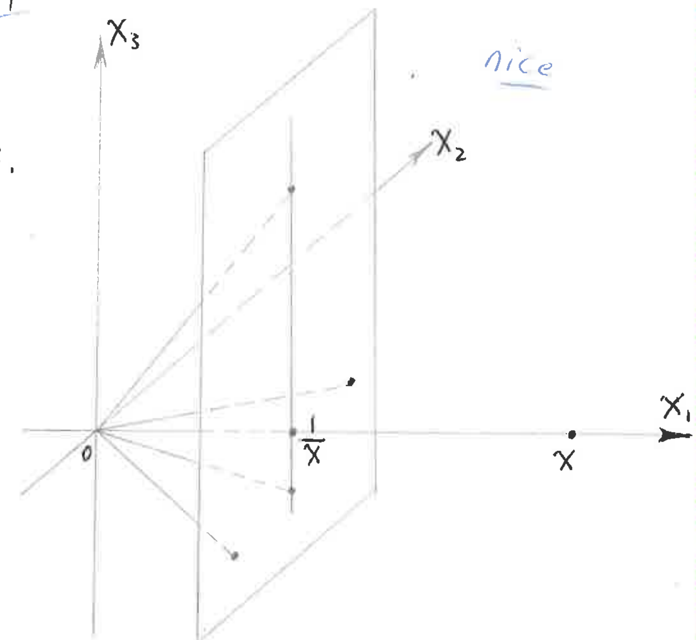
Take $(x_1, x_2, 0)$ in R^3. Take $y = (y_1, \frac{1-x_1y_1}{x_2}, y_3)$

$\langle x, y\rangle = x_1y_1 + x_2\frac{1-x_1y_1}{x_2} + 0 \cdot y_3 = 1$

y_1, y_3 can take any value. So

So there is more than one y.

nice



good

3. Suppose $\{x^1, x^2, \ldots, x^m\}$ is linearly dependent, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, such that

$$\alpha_1 x^1 + \alpha_2 x^2 + \cdots + \alpha_m x^m = 0$$

assuming $\alpha_1 \neq 0$ o.k.

Therefore
$$x^1 = -\frac{\alpha_2}{\alpha_1}x^2 - \frac{\alpha_3}{\alpha_1}x^3 - \frac{\alpha_4}{\alpha_1}x^4 - \cdots - \frac{\alpha_m}{\alpha_1}x^m$$

Then

$\langle x^1, x^2\rangle = \langle -\frac{\alpha_2}{\alpha_1}x^2 - \frac{\alpha_3}{\alpha_1}x^3 - \cdots - \frac{\alpha_m}{\alpha_1}x^m, x^2\rangle = -\frac{\alpha_2}{\alpha_1}\langle x^2, x^2\rangle - \frac{\alpha_3}{\alpha_1}\langle x^3, x^2\rangle - \cdots - \frac{\alpha_m}{\alpha_1}\langle x^m, x^2\rangle$

Since for all $x^j \neq x^i$, $\langle x^j, x^i\rangle = 0$

$\langle x^1, x^2\rangle = -\frac{\alpha_2}{\alpha_1}\langle x^2, x^2\rangle = -\frac{\alpha_2}{\alpha_1}\|x^2\|^2$

$x^2 \neq 0$, if $\alpha_2 \neq 0$, $\langle x^1, x^2\rangle \neq 0$, a contradiction, so $\{x^1, x^2, \ldots, x^m\}$ is linearly independent.

If $\alpha_2 = 0$, there exists at least one $\alpha_k \neq 0$. Then

$\langle x^1, x^k\rangle = -\frac{\alpha_k}{\alpha_1}\langle x^k, x^k\rangle = -\frac{\alpha_k}{\alpha_1}\|x^k\|^2 \neq 0$, a contradition.

4. $\langle x, y - \frac{\langle x,y\rangle}{\langle x,x\rangle}x\rangle = \langle x, y\rangle + \langle x, -\frac{\langle x,y\rangle}{\langle x,x\rangle}x\rangle = \langle x,y\rangle + (-\frac{\langle x,y\rangle}{\langle x,x\rangle})\langle x,x\rangle = \langle x,y\rangle - \langle x,y\rangle$

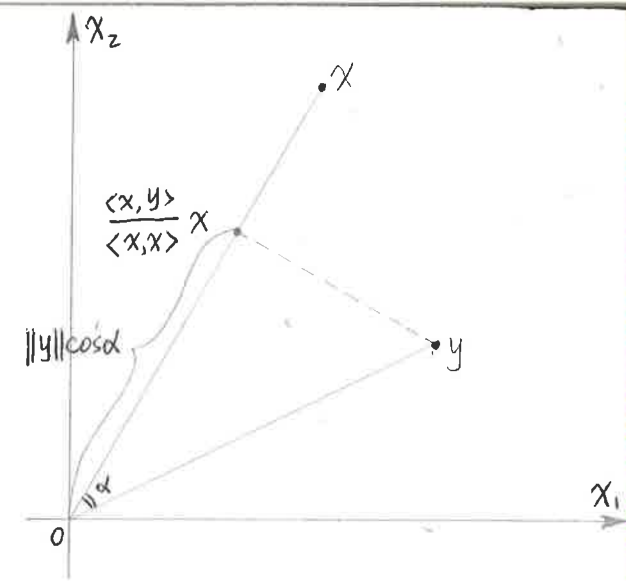
$= 0,$

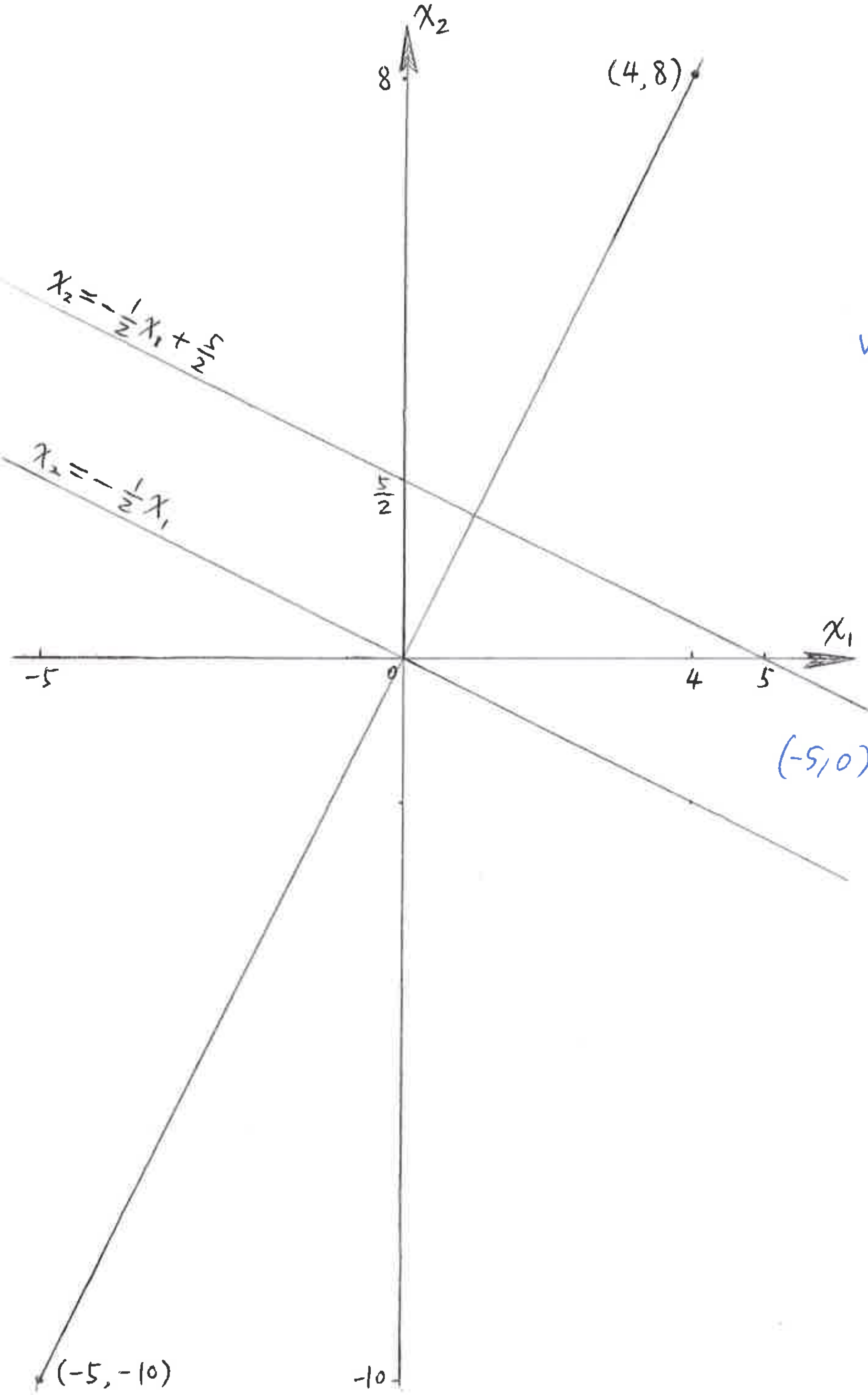
So x and $y - \frac{\langle x,y\rangle}{\langle x,x\rangle}x$ are orthogonal.

Geometrically, $\frac{\langle x,y\rangle}{\langle x,x\rangle}x$ has the same direction as x and its norm

$$\left\|\frac{\langle x,y\rangle}{\langle x,x\rangle}x\right\|=\left\|\frac{\|x\|\|y\|\cos\alpha}{\|x\|^2}x\right\|=\|y\|\cos\alpha$$

As shown in the graph on the right.





✓ 5. The corresponding homogeneous system of equations should be

$$\begin{cases}4x_1+8x_2=0\\-5x_1-10x_2=0\end{cases}$$

The solutions are on the line $x_2=-\frac{1}{2}x_1$ as shown on the graph on the left.

One of the solutions to the original system $\begin{cases}4x_1+8x_2=-20\\-5x_1-10x_2=25\end{cases}$

(-5,0)

is (5,0), So the solutions to the original system are on the line $x_2=-\frac{1}{2}x_1+\frac{5}{2}$

The "row vectors" (4,8) and (−5,−10) are orthogonal to the solutions of the homogeneous system. Because:

Take any homogeneous system

$$\begin{cases}a_1x_1+a_2x_2=0\\b_1x_1+b_2x_2=0\end{cases}$$

The inner product of the "row vectors" and the solution (x_1,x_2) should be

$$\langle(a_1,a_2)\cdot(x_1,x_2)\rangle=a_1x_1+a_2x_2=0$$
$$\langle(b_1,b_2)\cdot(x_1,x_2)\rangle=b_1x_1+b_2x_2=0$$

So they are orthogonal.

✓

6. Simon + Blume, Exercise 8.1.

a)

$$A+B=\begin{pmatrix}2&3&1\\0&-1&2\end{pmatrix}+\begin{pmatrix}0&1&-1\\4&-1&2\end{pmatrix}=\begin{pmatrix}2+0&3+1&1-1\\0+4&-1-1&2+2\end{pmatrix}=\begin{pmatrix}2&4&0\\4&-2&4\end{pmatrix}$$

A−D is not defined, since A is a 2×3 matrix while D is a 2×2 matrix.

$$3B=3\begin{pmatrix}0&1&-1\\4&-1&2\end{pmatrix}=\begin{pmatrix}0&3&-3\\12&-3&6\end{pmatrix}$$

$$DC=\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}1&2\\3&-1\end{pmatrix}=\begin{pmatrix}2\times1+1\times3&2\times2-1\times1\\1\times1+1\times3&2\times1-1\times1\end{pmatrix}=\begin{pmatrix}5&3\\4&1\end{pmatrix}$$

$$B^T=\begin{pmatrix}0&1&-1\\4&-1&2\end{pmatrix}^T=\begin{pmatrix}0&4\\1&-1\\-1&2\end{pmatrix}$$

$$A^TC^T=\begin{pmatrix}2&3&1\\0&-1&2\end{pmatrix}^T\begin{pmatrix}1&2\\3&-1\end{pmatrix}^T=\begin{pmatrix}2&0\\3&-1\\1&2\end{pmatrix}\begin{pmatrix}1&3\\2&-1\end{pmatrix}=\begin{pmatrix}2\times1+0\times2&2\times3+0\times(-1)\\3\times1-1\times2&3\times3-1\times(-1)\\1\times1+2\times2&1\times3+2\times(-1)\end{pmatrix}=\begin{pmatrix}2&6\\1&10\\5&1\end{pmatrix}$$

$$C+D=\begin{pmatrix}1&2\\3&-1\end{pmatrix}+\begin{pmatrix}2&1\\1&1\end{pmatrix}=\begin{pmatrix}1+2&2+1\\3+1&-1+1\end{pmatrix}=\begin{pmatrix}3&3\\4&0\end{pmatrix}$$

$$B-A=\begin{pmatrix}0&1&-1\\4&-1&2\end{pmatrix}-\begin{pmatrix}2&3&1\\0&-1&2\end{pmatrix}=\begin{pmatrix}0-2&1-3&-1-1\\4-0&-1+1&2-2\end{pmatrix}=\begin{pmatrix}-2&-2&-2\\4&0&0\end{pmatrix}$$

AB is not defined, as A has 3 columns while B has only 2 rows.

$$CE=\begin{pmatrix}1&2\\3&-1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1\times1+2\times(-1)\\3\times1-1\times(-1)\end{pmatrix}=\begin{pmatrix}-1\\4\end{pmatrix}$$

$$-D=-\begin{pmatrix}2&1\\1&1\end{pmatrix}=\begin{pmatrix}-2&-1\\-1&-1\end{pmatrix}$$

$$(CE)^T=\left(\begin{pmatrix}1&2\\3&-1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}\right)^T=\begin{pmatrix}1\times1+2\times(-1)\\3\times1-1\times(-1)\end{pmatrix}^T=\begin{pmatrix}-1\\4\end{pmatrix}^T=\begin{pmatrix}-1&4\end{pmatrix}$$

B+C is not defined since B is a 2×3 matrix while C is a 2×2 matrix.

$$D-C=\begin{pmatrix}2&1\\1&1\end{pmatrix}-\begin{pmatrix}1&2\\3&-1\end{pmatrix}=\begin{pmatrix}2-1&1-2\\1-3&1-(-1)\end{pmatrix}=\begin{pmatrix}1&-1\\-2&2\end{pmatrix}$$

$$CA=\begin{pmatrix}1&2\\3&-1\end{pmatrix}\begin{pmatrix}2&3&1\\0&-1&2\end{pmatrix}=\begin{pmatrix}1\times2+2\times0&1\times3+2\times(-1)&1\times1+2\times2\\3\times2+(-1)\times0&3\times3+(-1)\times(-1)&3\times1+(-1)\times2\end{pmatrix}=\begin{pmatrix}2&1&5\\6&10&1\end{pmatrix}$$

EC is not defined, as E has 1 columns while C has 2 rows.

$$(CA)^T=\left(\begin{pmatrix}1&2\\3&-1\end{pmatrix}\begin{pmatrix}2&3&1\\0&-1&2\end{pmatrix}\right)^T=\begin{pmatrix}1\times2+2\times0&1\times3+2\times(-1)&1\times1+2\times2\\3\times2+(-1)\times0&3\times3+(-1)\times(-1)&3\times1+(-1)\times2\end{pmatrix}^T=\begin{pmatrix}2&1&5\\6&10&1\end{pmatrix}^T=\begin{pmatrix}2&6\\1&10\\5&1\end{pmatrix}$$

$$E^TC^T=\begin{pmatrix}1\\-1\end{pmatrix}^T\begin{pmatrix}1&2\\3&-1\end{pmatrix}^T=(1,\ -1)\begin{pmatrix}1&3\\2&-1\end{pmatrix}=\begin{pmatrix}1\times1-1\times2&1\times3-1\times(-1)\end{pmatrix}=\begin{pmatrix}-1&4\end{pmatrix}$$

b) $$(DA)^T=\left(\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}2&3&1\\0&-1&2\end{pmatrix}\right)^T=\begin{pmatrix}2\times2+1\times0&2\times3+1\times(-1)&2\times1+1\times2\\1\times2+1\times0&1\times3+1\times(-1)&1\times1+1\times2\end{pmatrix}^T=\begin{pmatrix}4&5&4\\2&2&3\end{pmatrix}^T=\begin{pmatrix}4&2\\5&2\\4&3\end{pmatrix}$$

$$A^TD^T=\begin{pmatrix}2&3&1\\0&-1&2\end{pmatrix}^T\begin{pmatrix}2&1\\1&1\end{pmatrix}^T=\begin{pmatrix}2&0\\3&-1\\1&2\end{pmatrix}\begin{pmatrix}2&1\\1&1\end{pmatrix}=\begin{pmatrix}2\times2+0\times1&2\times1+0\times1\\3\times2+(-1)\times1&3\times1+(-1)\times1\\1\times2+2\times1&1\times1+2\times1\end{pmatrix}=\begin{pmatrix}4&2\\5&2\\4&3\end{pmatrix}$$

So $(DA)^T=A^TD^T$

c) $$CD=\begin{pmatrix}1&2\\3&-1\end{pmatrix}\begin{pmatrix}2&1\\1&1\end{pmatrix}=\begin{pmatrix}1\times2+2\times1&1\times1+2\times1\\3\times2-1\times1&3\times1-1\times1\end{pmatrix}=\begin{pmatrix}4&3\\5&2\end{pmatrix}$$

$$DC=\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}1&2\\3&-1\end{pmatrix}=\begin{pmatrix}2\times1+1\times3&2\times2-1\times1\\1\times1+1\times3&2\times1-1\times1\end{pmatrix}=\begin{pmatrix}5&3\\4&1\end{pmatrix}$$

So $CD\neq DC$

7. Let $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ $AE=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33}\end{pmatrix}$

So AE is just the result of interchanging the first two columns of A.

$$EA=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}=\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

So EA is just the result of interchanging the first two rows of A.

8.15 Check $\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2\end{pmatrix}=\begin{pmatrix} 2\times1+1\times(-1) & 2\times(-1)+1\times2 \\ 1\times1+1\times(-1) & 1\times(-1)+1\times2\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}=I$, So $\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}^{-1}=\begin{pmatrix} 1 & -1 \\ -1 & 2\end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix} .5 & 0 & -.5 \\ .5 & 0 & .5 \\ -.5 & 1 & -.5\end{pmatrix}=\begin{pmatrix} 1\times.5+1\times.5 & 0\times1 & 1\times(-.5)+1\times.5 \\ 1\times.5-1\times.5 & 1\times1 & 1\times.5-1\times(-.5) \\ -1\times.5+1\times.5 & 1\times0 & -1\times(-.5)+1\times.5\end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}=I, \text{ So} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0\end{pmatrix}^{-1}=\begin{pmatrix} .5 & 0 & -.5 \\ .5 & 0 & .5 \\ -.5 & 1 & -.5\end{pmatrix}$$

We can also check $\begin{pmatrix} 1 & -1 \\ -1 & 2\end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}=I$ and $\begin{pmatrix} .5 & 0 & -.5 \\ .5 & 0 & .5 \\ -.5 & 1 & -.5\end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0\end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}=I$.

8.28. The inverse of D is $\begin{pmatrix} d_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & d_2^{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n^{-1}\end{pmatrix}$. Check $\begin{pmatrix} d_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & d_2^{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n^{-1}\end{pmatrix}\begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n\end{pmatrix}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1\end{pmatrix}$

Also $\begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n\end{pmatrix}\begin{pmatrix} d_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & d_2^{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n^{-1}\end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}=I$

9.13 a) $A=\begin{pmatrix} 5 & 1 \\ 2 & -1\end{pmatrix}$, $b=\begin{pmatrix} 3 \\ 4\end{pmatrix}$. So $x_1=\dfrac{\begin{vmatrix} 3 & 1 \\ 4 & -1\end{vmatrix}}{\begin{vmatrix} 5 & 1 \\ 2 & -1\end{vmatrix}}=\dfrac{3\times(-1)-1\times4}{5\times(-1)-1\times2}=1$, $x_2=\dfrac{\begin{vmatrix} 5 & 3 \\ 2 & 4\end{vmatrix}}{\begin{vmatrix} 5 & 1 \\ 2 & -1\end{vmatrix}}=\dfrac{5\times4-3\times2}{5\times(-1)-1\times2}=-2$. So $x=(1,-2)$

b) $A=\begin{pmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0\end{pmatrix}$, $b=\begin{pmatrix} 2 \\ 7 \\ 1\end{pmatrix}$. So $x_1=\dfrac{\begin{vmatrix} 2 & -3 & 0 \\ 7 & -6 & 1 \\ 1 & 10 & 0\end{vmatrix}}{\begin{vmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0\end{vmatrix}}=\dfrac{0+0+(-3)\times1\times1-0-10\times1\times2-0}{0+0+(-3)\times1\times1-0-10\times1\times2-0}=1$

$x_2=\dfrac{\begin{vmatrix} 2 & 2 & 0 \\ 4 & 7 & 1 \\ 1 & 1 & 0\end{vmatrix}}{\begin{vmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0\end{vmatrix}}=\dfrac{0+0+2\times1\times1-0-1\times1\times2-0}{0+0+(-3)\times1\times1-0-10\times1\times2-0}=0$, $x_3=\dfrac{\begin{vmatrix} 2 & -3 & 2 \\ 4 & -6 & 7 \\ 1 & 10 & 1\end{vmatrix}}{\begin{vmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0\end{vmatrix}}=\dfrac{-12+80-21+12-140+12}{-23}=3$

So $x=(1,0,3)$ is the unique solution.

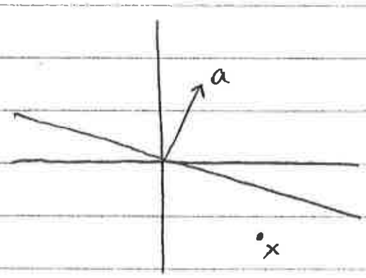
PSC 403: HMWK #5
(Due 12-9-97)

This homework is meant to help you review the material of the course, helping prepare you for the final. It is not an example of what the final might look like – it's longer, for one thing, and the general level of difficulty is higher. I'm putting "*" signs next to the required problems. (Those are the only ones you have to submit answers for.) I count 12 of them: #'s 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 15, and 17.

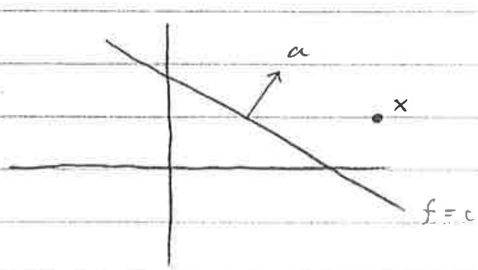
* 1. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be linear. If $f(1,2) = 3$ and $f(4,1) = -1$, what is the explicit form of $f(x_1, x_2)$? Use Cramer's rule P143 to answer this.

* 2. (a) Consider a linear function $f: \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x) = a \cdot x$, with level set through zero as below.

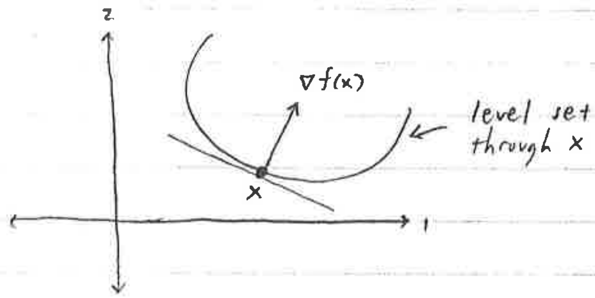
Consider point $x \in \mathbb{R}^2$ as indicated. Which case holds: $f(x) > 0$, $f(x) < 0$, or $f(x) = 0$? Why?



(b) Now consider the level set at c as below. Which case holds: $f(x) > c$, $f(x) < c$, or $f(x) = c$? Why?



* 3. We know $Df(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right)$. In class, I claimed that $\nabla f(x)$ is "orthogonal to" the level set of f through x:



Using the definition of the derivative, prove this. That is, prove: If $\langle x^k \rangle$ is a sequence of points on the level set through x eventually converging to x, then

$$\frac{\nabla f(x) \cdot (x^k - x)}{\|x^k - x\|} \to 0 .$$

* 4. Let there be three voters, 1, 2, and 3, and let policies be represented by real numbers. Voter i has preferences represented by $u_i(x) = -|x - x_i|$, where $x_1 = 0$, $x_2 = 3$, $x_3 = 4$.

(a) Graph these utility functions.

(b) Suppose two candidates are running for office. One's policy platform is denoted y and the other's is denoted z. If $y = 1$ and $z = 2$, and the voters vote for the candidates on the basis of these platforms in a majority rule election, which candidate wins?

(c) Suppose you are a candidate and care only about winning the election. (Tying is better than losing.) If you don't know your opponent's platform, what policy platform would you pick?

5. Consider the "lower triangular" matrix below:

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & & \vdots \\ a_{31} & a_{32} & a_{33} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Argue that $\det A = a_{11}a_{22}a_{33}\cdots a_{nn}$. (I don't need a formal proof of this, but a few lines of explanation.)

* 6. A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if

(i) $f(\alpha x) = \alpha f(x)$

(ii) $f(x+y) = f(x) + f(y)$.

(This definition looks just like the one we've seen, but now $f(x)$ is a vector in $\mathbb{R}^m$.)

(a) Given a $m \times n$ matrix A define $f: \mathbb{R}^n \to \mathbb{R}^m$ as follows:

$$f(x) = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

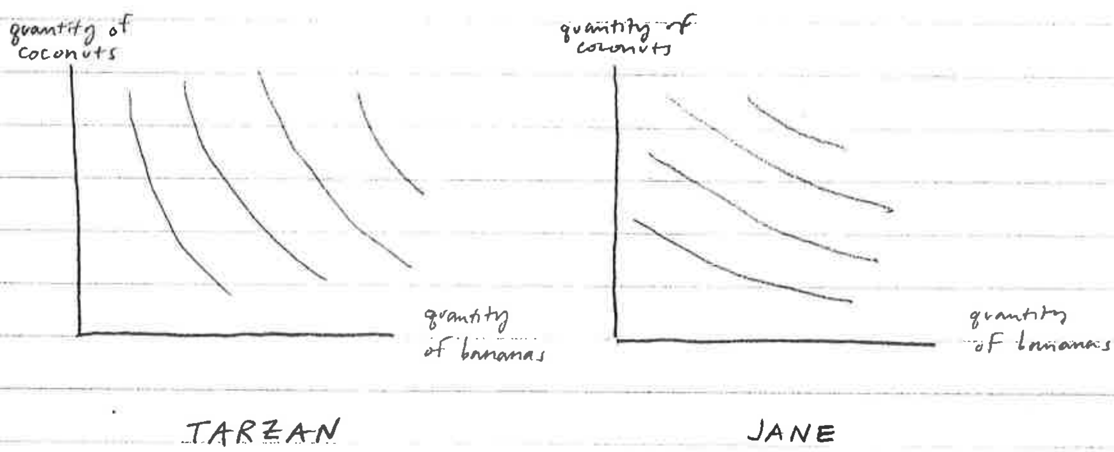
Prove that f is linear.

(b) Given a linear function $f: \mathbb{R}^n \to \mathbb{R}^m$, prove that there exists a $m \times n$ matrix A such that, for all $x \in \mathbb{R}^n$,

$$f(x) = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

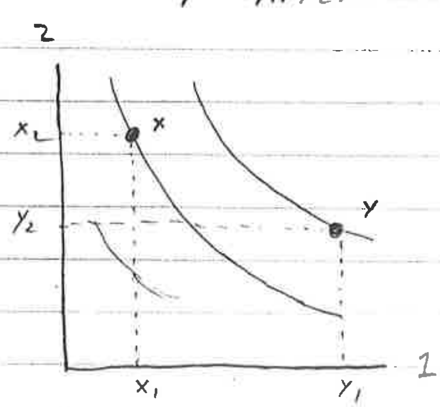
(Hint: The proof is very similar to the proof of the proposition on p. 155.)

* 7. Judging from the indifference curves below, who likes bananas (relative to coconuts) more: Tarzan or Jane?



Why? (Sorry about the dumb names.)

* 8. Now consider two consumers with identical preferences. (I'll only draw one set of indifference curves for the two.) Suppose consumer 1 has bundle $x = (x_1, x_2)$ and consumer 2 has bundle $y = (y_1, y_2)$. Is there any way for these two consumers to trade goods in a way that makes them both strictly better off? If so, draw their indifference curves and describe the trade graphically. (Note: after the trade the



total amount of good 1 consumed by the agents can't exceed $x_1 + y_1$, and the total amount of good 2 can't exceed $x_2 + y_2$.)

* 9. Now consider an individual who must choose some alternative(s) from a finite set X. I write xRy if x is at least as good as y, and I write xPy if x is better than y. Assume that, for all x and y, ① $xPy \Rightarrow xRy$ and ② $xRy \Leftrightarrow \neg yPx$.

We'll say R is acyclic if there do not exist a natural number K and alternatives $x_1, x_2 \ldots x_K$ such that

$$x_1 P x_2 P x_3 \cdots x_{K-1} P x_K P x_1.$$

We say an alternative x is choosable if, for all $y \in X$, xRy. Prove: If R is acyclic then there exists a choosable alternative.

10. Suppose there are three voters voting over three alternatives, a, b, and c. Give this small society preferences of majority rule. That is, alternative x is better than y for society iff two or more voters prefer x to y. Write xPy when this is the case.

(a) Assuming ② above, when is it the case that xRy.

(b) Suppose individual preferences over a, b, and c are given below, where R_i represents i's ordering and x appearing above y means x is better than y for the voter.

R_1	R_2	R_3
a	b	c
b	c	a
c	a	b

Are society's preferences acyclic?

* 11. Check to see whether the following sequences converge. If so, give the limit of the sequence.

(a) $\left(\frac{1}{n}, -3^n, \frac{n^2+3n-2}{n^2}\right)$

(b) $\left(n^{-2}\cos 3n, 1-\frac{1}{n}\right)$

(c) $\left(\frac{1}{n}, \log n, 6\right)$

* 12. Define $u: \mathbb{R}^3 \to \mathbb{R}$ by $u(x_1, x_2, x_3) = x_2^2 x_3 - \cos(x_1 + x_2)$.

(a) Calculate the partial derivatives of u.

(b) What is the directional derivative at $(\pi/3, 2\pi/3, 2)$ in the direction $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$?

(c) What is the marginal rate of substitution of 3 with respect to 2 at this point?

13. Consider $f: \mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$, and a sequence $\langle h_n \rangle$ eventually converging to 0. Suppose

$$\lim_{n\to\infty} \frac{|f(x+h_n) - f(x) - ah_n|}{|h_n|} = 0$$

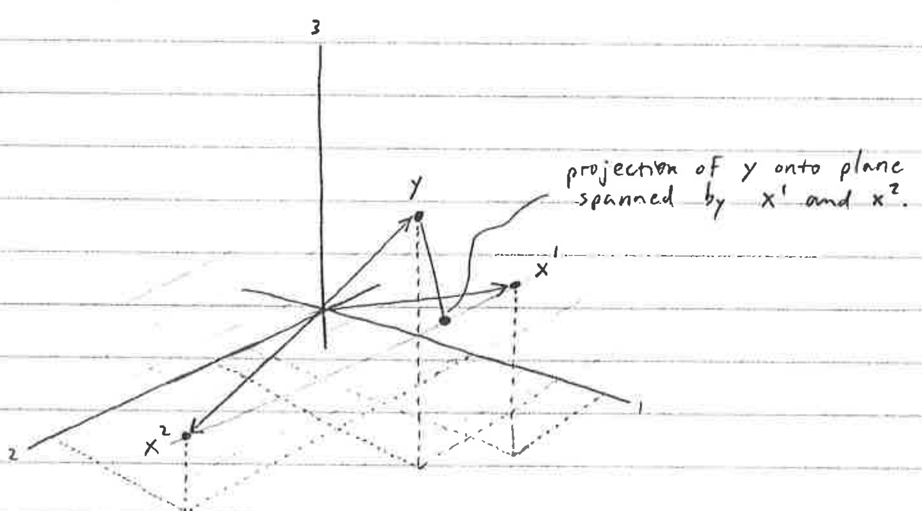
and

$$\lim_{n\to\infty} \frac{|f(x+h_n) - f(x) - bh_n|}{|h_n|} = 0,$$

where $a, b \in \mathbb{R}$. Show $a = b$.

This problem and the next are long, but they give some theory behind data analysis.

14. Let $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ be linearly independent, and let $S = \operatorname{span}\{x^1 \ldots x^k\}$ be the subspace of $\mathbb{R}^n$ with basis $\{x^1, \ldots, x^k\}$. Let $y \in \mathbb{R}^n$. We can find the point in S closest to y by "projecting" y onto S. When $n=3$ and $k=2$, this is depicted as follows.



Call this projection z. Since it is in S, there exist $\beta_1 \cdots \beta_k$ such that $z = \beta_1 x^1 + \cdots + \beta_k x^k$. Because z is a "projection," the vector $y - z$ should be orthogonal to $x^1, x^2, \ldots x^k$. That is, $x^j \cdot (y - z) = 0$ for $j = 1 \ldots k$.

Let's rewrite this in matrix notation. Let $x^j = \begin{pmatrix} x_1^j \\ \vdots \\ x_n^j \end{pmatrix}$ be a column vector, and form the $n \times k$ matrix X as

$$X = (x^1 \; x^2 \; \cdots \; x^k).$$

Letting $b = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$, we have

$$x^j \cdot (y - Xb) = 0 \quad \text{for} \quad j = 1 \ldots k,$$

where this is the usual dot product between two vectors. If I want to use matrix multiplication, essentially the same thing, I just need to write x^j as a row vector, to make sure matrix multiplication is well-defined.

$$\underbrace{(x_1^j \; x_2^j \; \cdots \; x_n^j)}_{\substack{1 \times n \\ \text{matrix}}} \underbrace{\begin{pmatrix} y - Xb \end{pmatrix}}_{\substack{n \times 1 \\ \text{matrix}}} = 0 \qquad j = 1 \ldots k$$

I can rewrite this as follows:

$$\underbrace{\begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_n^1 \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & & \\ x_1^k & x_2^k & \cdots & x_n^k \end{pmatrix}}_{k \times n} \underbrace{\begin{pmatrix} y - Xb \end{pmatrix}}_{n \times 1} = 0.$$

Note that the matrix on the left is just the transpose of X. Thus (Finally), we have

$$X^T(y - Xb) = 0.$$

Assuming X^TX is an invertible matrix, solve this equation for b in terms of the matrices X^T, y, and X. The solution tells us what z is.

The expression for b you derive is the formula for ordinary least squares, something you will see more of next term.

14. Suppose you conjecture that government spending is a linear function of GNP and the annual inflation rate, i.e.,

$$GOV = \beta_1 \, GNP + \beta_2 \, INF + \text{random error}$$

for some $\beta_1, \beta_2 \in \mathbb{R}$, but you don't know what those coefficients are. It makes sense to look at historical data on these variables to try and make some inference about β_1 and β_2. Let's use "observations" of these variables for the last three years:

	GOV	GNP	INF
1996	2.2	5.4	6
1995	2.4	5.2	3
1994	2.1	5.0	4

where GOV and GNP are in trillions of dollars and INF is in percent. (Warning: I've made these numbers up! I have no idea how close they are to reality.)

We want to find β_1 and β_2 that give us the closest match with reality. Letting,

$$y = \begin{pmatrix} 2.2 \\ 2.4 \\ 2.1 \end{pmatrix} \qquad x^1 = \begin{pmatrix} 5.4 \\ 5.2 \\ 5.0 \end{pmatrix} \qquad x^2 = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix},$$

we want to project y onto the subspace (a plane, in this case) spanned by x^1 and x^2. This will give us a vector $\beta_1 x^1 + \beta_2 x^2$, the best approximation of y we can get, given the data. Letting

$$X = \begin{pmatrix} 5.4 & 6 \\ 5.2 & 3 \\ 5.0 & 4 \end{pmatrix},$$

use your answer to problem 13 to find $b = (\beta_1, \beta_2)$. (You'll have to use Cramer's rule to get your solution.)

Congratulations! You've just done ordinary least squares estimation.

* 15. Which of the following sets are open? Which are closed? Fix $p \in \mathbb{R}^n$.

(a) $\{x \in \mathbb{R}^n \mid p \cdot x \geq 2\}$

(b) $\{x \in \mathbb{R}^n \mid p \cdot x < 3\}$

(c) $\{x \in \mathbb{R}^n \mid 2 \leq p \cdot x < 3\}$

(d) $\{x \in \mathbb{R}^n \mid x_1 \cdot x_2 \cdot \cdots \cdot x_n \geq 0\}$

(e) $\{x \in \mathbb{R}^n \mid 0 < x_j < 1, \ j = 1 \ldots n\}$

16. Use the definitions of directional derivative and derivative to prove that

$$D_t f(x) = Df(x) \cdot t .$$

(Hint: By definition of the directional derivative,

$$\lim_{\alpha_k \to 0} \frac{| f(x+\alpha_k t) - f(x) - D_t f(x) \alpha_k |}{|\alpha_k|} = 0 .$$

Defining $h_k = \alpha_k t$, it is clear that $h_k \to 0$. By the definition of the derivative,

$$\lim_{h_k \to 0} \frac{| f(x+h_k) - f(x) - Df(x) \cdot h_k |}{\| h_k \|} = 0 .$$

There's just one more step...)

* 17. Suppose there are two goods and a consumer has utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ defined as $u(x_1, x_2) = 3x_1 + 2x_2$. He/she has income I and prices of the two goods are p_1 and p_2, respectively. Describe the bundle(s) demanded by the consumer. (Of course, these will depend on p_1, p_2, and I, somehow...)

I've assigned some partial derivative problems, but you're encouraged to look over Simon + Blume's Section 14.1 and to do their problems 14.1 and 14.2. Sections 14.2 and 14.3 are also relevant.

December 13, 1997

PSC 403: MATHEMATICAL MODELLING
FINAL EXAM

The exam, worth a total of 180 points, is divided into three parts: definitions, calculation problems, and proofs. The weights of the different sections are listed below and indicate (what I think would be) reasonable allotments of time to them. The problems in a given section will have equal weight.

One notational reminder: I use $\Re_+$ to denote the set of non-negative real numbers and $\Re^n_+$ to denote the set of vectors in $\Re^n$ with non-negative coordinates.

If you get stuck on one part of a question, move on and try to finish as much of the question as you can. It goes without saying, I suppose, but show all your work! And good luck!

1. Definitions. (25pt.s) Give precise mathematical descriptions of the following concepts and results.

1.1 Let f denote a function mapping a set X to another set Y. What does it mean for f to be 1-1?

1.2 What does it mean for a subset $X \subseteq \Re$ to be compact?

1.3 Let $f : \Re \to \Re$. What does it mean for f to be concave?

1.4 State the Mean Value Theorem.

1.5 Given $f : \Re^n \to \Re$, what does it mean for f to be linear?

1.6 Given a vector $x \in \Re^n$, what is the definition of $||x||$? How do we define $B_\epsilon(x)$?

1.7 What does it mean for a sequence $\langle x^k \rangle$ of vectors in $\Re^n$ to converge to a vector x?

1.8 Given $f : \Re^n \to \Re$ and a vector $x \in \Re^n$, what does it mean for f to be differentiable at x?

2. Calculations. (80pt.s) Complete all the problems of this section.

2.1 Define the following matrices.

$$A = \begin{pmatrix} 3 & -1 & 2 & 0 \\ 1 & 0 & -4 & 4 \\ -2 & 1 & 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 1 & 4 \\ -2 & -2 & 2 \\ 0 & 1 & 3 \end{pmatrix}.$$

What are AB and $|AB|$? What do you conclude about the rank of AB?

2.2 (a) The uniform distribution is given by the density function,

$$g(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise,} \end{cases}$$

where a and b are real numbers with $a < b$. Calculate the expected value of a uniformly distributed random variable: that is, find

$$\int_{-\infty}^{\infty} tg(t)\, dt.$$

(b) Recall the exponential distribution is given by the density function,

$$f(x) = \begin{cases} \frac{1}{\lambda}e^{-x/\lambda} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

where λ is some fixed positive real number. Calculate the expected value of an exponentially distributed random variable: that is, find

$$\int_{-\infty}^{\infty} tf(t)\, dt.$$

(You will need to use L'Hôpital's rule: given two differentiable functions of one variable, say F and G,

$$\lim_{x\to\infty} \frac{F(x)}{G(x)} = \lim_{x\to\infty} \frac{F'(x)}{G'(x)}.)$$

2.3 Consider the problem of maximizing or minimizing $f(x, y) = -x^2 - y^2$ subject to the constraint that $x+y = 1$. Set up the Lagrangian for this problem, and find the critical point of the Lagrangian. We didn't talk about second order conditions for constrained optimization problems, but can you tell whether this point is a global maximizer or minimizer of f subject to $x + y = 1$?

2.4 Consider a community of 100 voters, numbered from 1 to 100. Voters consume just two goods: a commonly enjoyed good (like police protection), denoted x, and a privately consumed good, denoted y. To denote a specific voter's consumption of the private good, say the ith voter, we add the subscript i to y, like this: y_i. Each voter has an income of 1000 dollars.

Voter i's preferences over (x, y_i) bundles are represented by a utility function $u_i(x, y_i) = x^{\alpha_i} + y_i$, where $0 < \alpha_i < 1$. The commonly consumed good, x, costs one dollar per unit, which the community must raise by taxing its members. Let's say the cost of providing x units is divided evenly: each consumer i pays $\frac{x}{100}$ dollars in taxes and can spend her remaining income on the private good. That is, if x units are provided by the community, voter i consumes $y_i = 1000 - \frac{x}{100}$ units of the private good.

(a) Substitute the above expression for y_i into voter i's utility function to get her utility in terms of x alone. Verify that this function of x is concave on $\Re_+$.

(b) Calculate voter i's ideal level of the commonly consumed good (meaning, the amount of x that the voter would choose for the community, if she had her way).

To make things simple, let's assume there are only three types of voters: those with $\alpha_i = 1/2$, those with $\alpha_i = 1/3$, and those with $\alpha_i = 1/4$. There are 45, 30, and 25 voters of each type, respectively.

Imagine that the community holds a majority rule election to decide how much of the commonly consumed good to buy. There two candidates, who only care about winning the election, and each publicly commits to a level of the good that he will purchase for the community if elected. Once the candidates have chosen their positions, the voters cast their ballots for the candidate whose position they like most.

(c) How much of the commonly consumed good would you expect to be provided?

3. Proofs. (75pt.s) Complete all of the problems in this section.

3.1 Prove by induction that $n^2 = \sum_{k=1}^{n}(2k-1)$.

3.2 Let $f : \Re \to \Re$ satisfy the following condition. There exists a number $c > 0$ such that, for all $x, y \in \Re$, $|f(x) - f(y)| < c|x - y|$. Prove that f is continuous. How would things change if f were a function of n variables rather than just one?

3.3 Consider the following system of linear equations.

$$\begin{aligned} \alpha_{11}x_1 + \cdots + \alpha_{1n}x_n &= y_1 \\ \vdots \quad\quad & \vdots \quad \vdots \\ \alpha_{n1}x_1 + \cdots + \alpha_{nn}x_n &= y_n \end{aligned}$$

Argue that if the system always has a solution (meaning, it has a solution no matter what $y_1, \ldots, y_n$ are), then it always has a *unique* solution. How would you go about finding it?

3.4 Consider three vectors $x, y, z \in \Re^n$ and any $\epsilon > 0$. Prove that $x \in B_{\epsilon/2}(z)$ and $y \in B_{\epsilon/2}(x)$ imply $y \in B_\epsilon(z)$. (Drawing a picture for the $n = 2$ case may help.)

3.5 Assume $f : \Re^n \to \Re$ is differentiable at x. Prove that the slope of f at x in the direction of steepest ascent is $||\nabla f(x)||$.

THE END. Congratulations on finishing the test, and have a great winter break!