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Tutorial 3:
MORE EXAMPLES ON THE DERIVATION OF THE DIRECT INTEGRAL FORMS

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Summary and objectives

In tutorial 2, we demonstrated the basic steps for deriving the direct boundary integral equation. In this tutorial we will explain, with more examples, the derivation of the direct boundary integral equation for systems of partial differential equations. We will consider two examples: elasticity problems and shear-deformable plate bending problems.

1 Elasticity equations

In this section, we will derive the direct boundary integral equation form for elasticity problems (see Ref. [1] for more details about the theory).

1.1 Governing equations

$$\sigma_{ij,j} + b_i = 0 \tag{1}$$

where (σ) denotes the stresses and (b) are the body forces. Equation (1) is called the Navier equation in terms of stresses.

1.2 Problem variables

u_i: displacements, p_j: tractions, where

$$p_i = \sigma_{ij} n_j \tag{2}$$

in which (n) is the normal to the boundary (Γ).

1.3 Strain-displacement relationships

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

and

$$\varepsilon_{ii} = u_{i,i} \tag{4}$$

where (ε) denotes the strains.

1.4 Stress-displacement relationships (Hooke's law)

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} + \varepsilon_{ij}\right]$$
$$= \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\delta_{ij}u_{k,k} + \frac{u_{i,j}+u_{j,i}}{2}\right] \qquad (5)$$

in which (E) is the elasticity modulus (the Young's modulus) and (ν) is the Poisson's ratio.

1.5 Integral representation

By weighting equation (1) using certain functions (U_i) and integrating it over the problem domain (Ω), the following integral identity can be established:

$$\int_\Omega \left(\sigma_{ij,j} + b_i\right) U_i \, d\Omega = 0 \qquad (6)$$

It has to be noted that equation (3) is a scalar equation (all involved indexes are dummy). Applying the integration by parts (Green's identity) on the first term in equation (6), it gives:

$$\int_\Gamma U_i\left(\sigma_{ij}n_j\right)d\Gamma - \int_\Omega \sigma_{ij}U_{i,j}\, d\Omega + \int_\Omega b_iU_i\, d\Omega = 0 \qquad (7)$$

Replacing the stresses with the tractions (equation (2)) in the first integral and with the displacement derivatives from equation (5) in the second integral, it gives:

$$\int_\Gamma U_ip_i\, d\Gamma - \int_\Omega U_{i,j}\frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\delta_{ij}u_{k,k} + \frac{u_{i,j}+u_{j,i}}{2}\right]d\Omega + \int_\Omega b_iU_i\, d\Omega = 0 \qquad (8)$$

Applying the integration by parts for the second integral, one obtains:

$$\int_\Gamma U_ip_i\, d\Gamma + \int_\Omega b_iU_i\, d\Omega - \int_\Gamma U_{i,j}\frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\delta_{ij}u_kn_k + \frac{u_in_j+u_jn_i}{2}\right]d\Gamma$$
$$+ \int_\Omega \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\delta_{ij}u_kU_{i,jk} + \frac{u_iU_{i,jj}+u_jU_{i,ji}}{2}\right]d\Omega = 0 \qquad (9)$$

Replace (u_k) by ($u_i\ \delta_{ki}$) in the third and the fourth integrals and (u_j) by ($u_i\ \delta_{ij}$) and group together terms, to obtain:

$$\int_\Gamma U_ip_i\, d\Gamma + \int_\Omega b_iU_i\, d\Omega - \int_\Gamma u_i\frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\delta_{ij}\overbrace{U_{i,j}\delta_{ki}n_k}^{\text{term1}} + \frac{U_{i,j}n_j + \overbrace{U_{i,j}\delta_{ij}n_i}^{\text{term2}}}{2}\right]d\Gamma$$

$$+\int_{\Omega} u_i \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\delta_{ij}\overbrace{U_{i},_{jk}\delta_{ki}}^{term3}+\frac{U_{i},_{jj}+\overbrace{U_{i},_{ji}\delta_{ij}}^{term4}}{2}\right]d\Omega \tag{10}$$

Now consider the terms shown in the above equation:

<u>term1:</u> $(U_{i},_{j}\ \delta_{ki}\ n_k) \rightarrow (U_{k},_{k}\ n_j)$ Note: (k) is a dummy index

<u>term2:</u> $(U_{i},_{j}\ \delta_{ij}\ n_i) \rightarrow (U_{j},_{i}\ n_j)$

<u>term3:</u> $(\delta_{ki}\ U_{i},_{jk}) \rightarrow (U_{k},_{kj})$

<u>term4:</u> $(\delta_{ij}\ U_{i},_{ji}) \rightarrow (U_{j},_{ij})$

Then the integral equation can be re-written as follows:

$$\int_{\Gamma} U_i p_i\, d\Gamma + \int_{\Omega} b_i U_i\, d\Omega - \int_{\Gamma} u_i \overbrace{\frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\delta_{ij}U_{k},_{k}+\frac{U_{i},_{j}+U_{j},_{i}}{2}\right]}^{\boldsymbol{\sigma}_{ij}} n_j\, d\Gamma$$

$$+\int_{\Omega} u_i \underbrace{\frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\delta_{ij}U_{i},_{jk}\delta_{ki}+\frac{U_{i},_{jj}+U_{i},_{ji}\delta_{ij}}{2}\right]}_{\boldsymbol{\sigma}_{ij},_{j}} d\Omega = 0 \tag{11}$$

Where the bold ($\boldsymbol{\sigma}$) denotes the stresses corresponding to the weighting functions (U). From equation (2), the following traction-stresses equation can be written:

$$P_i = \boldsymbol{\sigma}_{ij} n_j \tag{12}$$

where the (P) denotes the tractions corresponding to the weighting functions (U). Then equation (11) can be re-written as follows:

$$\int_{\Gamma} U_i p_i\, d\Gamma - \int_{\Gamma} u_i P_i\, d\Gamma + \int_{\Omega} b_i U_i\, d\Omega + \int_{\Omega} u_i \boldsymbol{\sigma}_{ij},_{j}\, d\Omega = 0 \tag{13}$$

This equation represents the virtual work principal between two states (u, p) and (U, P). It is also called the Maxial-Betti reciprocal principal. The followings can be highlighted on this identity:

1- It is a scalar equation, i.e. all indexes are dummy indexes.
2- The function (U) can be chosen as any arbitrary function.
3- Two domain integrals appeared in this identity (recall tutorial 2), one for the body forces and the other where the differential operator is swapped from being applied on (u) to be applied on (U). The rest of the integrals are boundary terms.
4- It is defined at a point, therefore it is more convenient to write it for any point (ξ) inside the domain (Ω) as follows:

$$\int_{\Gamma(\xi)} U_i(\xi) p_i(\xi)\, d\Gamma(\xi) - \int_{\Gamma(\xi)} u_i(\xi) P_i(\xi)\, d\Gamma(\xi) + \int_{\Omega(\xi)} b_i(\xi) U_i(\xi)\, d\Omega(\xi)$$

$$+\int_{\Omega(\xi)} u_i(\xi)\boldsymbol{\sigma}_{ij},_j(\xi)\,d\Omega(\xi)=0 \quad (14)$$

The most important aspect now is how to choose the functions (U) in order to get rid of the last domain integral. To do so, one can choose the (U) as the displacements corresponding to single unit load in infinite domain, or:

$$\boldsymbol{\sigma}_{kij},_j(\xi,\mathbf{x})=-\delta(\xi,\mathbf{x})\delta_{ki} \quad (15)$$

where the $\delta(\xi,\mathbf{x})$ is the Dirac delta distribution. There are several items to highlight in this equation:

1- We introduce the point (**x**) to represent the field point; at which (U & P) will result due to an applied unit load at the source point (ξ).

2- We introduce an additional index (k), which represent the direction of the applied unit load at the point (ξ); therefore this index should appear in (U & P) from now on and henceforth. It has to be noted that the index (k) is the only non-dummy index in the equation.

3- From the properties of the Dirac delta distribution, the resulting (U & P) will no longer be functions, they will be two-point kernels and will be denoted as the "fundamental solution kernels".

By substituting using equation (15) into the last domain integral in equation (14) and making use of the properties of the Dirac delta, one obtains:

$$\int_{\Omega(\xi)} u_i(\xi)\left[-\delta(\xi,\mathbf{x})\delta_{ki}\right]d\Omega(\xi)=-u_k(\xi) \quad (16)$$

Substitute this result into equation (14), to give:

$$u_k(\xi)+\int_{\Gamma(\mathbf{x})}\left[U_{ki}(\xi,\mathbf{x})p_i(\mathbf{x})-P_{ki}(\xi,\mathbf{x})u_i(\mathbf{x})\right]d\Gamma(\mathbf{x})+\int_{\Omega(\mathbf{X})}U_{ki}(\xi,\mathbf{X})b_i(\mathbf{X})\,d\Omega(\mathbf{X})=0 \quad (17)$$

where (**x**) is a field point on the boundary, (**X**) is an internal field point and (ξ) is an internal source point. Equation (17) represents the Somigliana identity and it is the boundary integral equation for internal collocation point (ξ). The symbols (U_{ki} & P_{ki}) are the two-point fundamental solution kernels (known as Kelvin's fundamental solution; although some times they are also called free-space Green's functions). They represent the displacements and tractions at point (**x**) in the direction (i) due to applied unit load at the point (ξ) in the direction (k). It has to be noted that, in order to use the former integral identity inside the boundary element method, the point (ξ) has to be taken to the boundary. This will be discussed in a future tutorial.

2 Shear-deformable plate bending equations

The direct integral equations for the shear-deformable plate bending problems will be derived in this section as another example (for more details, see Ref. [2]).

2.1 Governing equations

$$M_{\alpha\beta},_\beta-Q_\alpha=0 \quad (18)$$

$$Q_{\alpha},_{\alpha}+q=0 \qquad (19)$$

where (M & Q) are the bending moment and shear forces stress resultants and (q) denotes the applied uniform domain loading. Noting that, in this section, the Roman indexes vary from 1 to 3 and the Greek indexes will vary from 1 to 2.

2.2 Problem variables

u_i: u_α (rotations) & u_3 (deflections): generalized displacements and p_j: generalized tractions, where

$$p_\alpha = M_{\alpha\beta} n_\beta \qquad (20)$$

$$p_3 = Q_\alpha n_\alpha \qquad (21)$$

in which the same notation is used as that of the former section.

2.3 Stress resultant-displacement relationships (the Generalized Hooke's law)

$$M_{\alpha\beta} = \frac{D(1-\nu)}{2}\left(u_{\alpha},_{\beta} + u_{\beta},_{\alpha} + \frac{2\nu}{1-\nu} u_{\gamma},_{\gamma}\, \delta_{\alpha\beta} \right) + \frac{\nu}{(1-\nu)\lambda^2} q \delta_{\alpha\beta} \qquad (22)$$

$$Q_\alpha = \frac{D(1-\nu)}{2} \lambda^2 \left(u_\alpha + u_3,_{\alpha} \right) \qquad (23)$$

in which (D) is the modulus of rigidity for the plate and (λ) is the shear factor. (q) is the applied uniform load of the domain.

2.4 Integral representation

By weighting equation (1) using certain functions (U_i: U_α & U_3) and integrating it over the problem domain, the following integral identity can be established:

$$\int_\Omega \left[\left(M_{\alpha\beta},_{\beta} - Q_\alpha \right) U_\alpha + \left(Q_\alpha,_{\alpha} + q \right) U_3 \right] d\Omega = 0 \qquad (24)$$

Applying the integration by parts for both terms, it gives:

$$\int_\Gamma U_\alpha M_{\alpha\beta} n_\beta d\Gamma - \int_\Omega M_{\alpha\beta} U_\alpha,_{\beta}\, d\Omega - \int_\Omega Q_\alpha U_\alpha d\Omega + \int_\Gamma U_3 Q_\alpha n_\alpha d\Gamma$$
$$- \int_\Omega Q_\alpha U_3,_{\alpha}\, d\Omega + \int_\Omega q U_3 d\Omega = 0 \qquad (25)$$

Now we are going to perform the following algebraic operations on the former equation:

1- Use equations (20) and (21) in the first and the fourth integrals respectively,
2- Group the third and the fifth integrals and replace the shear stress resultant by the equivalent generalized displacement derivatives from equation (23),
3- Replace the moment stress resultant in the second integral by the equivalent generalized displacement derivatives from equation (22).

After performing all the above operations, we obtain:

$$\int_{\Gamma} U_\alpha p_\alpha d\Gamma - \int_{\Omega} \overbrace{\left\{ \frac{D(1-\nu)}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha} + \frac{2\nu}{1-\nu} u_{\gamma,\gamma}\,\delta_{\alpha\beta} \right) + \frac{\nu}{(1-\nu)\lambda^2} q\delta_{\alpha\beta} \right\}}^{M_{\alpha\beta}} U_{\alpha,\beta}\, d\Omega$$

$$+ \int_{\Gamma} U_3 p_3 d\Gamma - \int_{\Omega} \underbrace{\frac{D(1-\nu)}{2}\lambda^2 \left(u_\alpha + u_{3,\alpha}\right)}_{Q_\alpha} \left(U_\alpha + U_{3,\alpha}\right) d\Omega + \int_{\Omega} qU_3 d\Omega = 0 \qquad (26)$$

This equation can be re-arranged as follows after applying the integration by parts another time on the second integral and grouping the first and the third integrals:

$$\int_{\Gamma} U_i p_i d\Gamma - \int_{\Omega} \mathbf{Q}_\alpha \left(u_\alpha + u_{3,\alpha}\right) d\Omega + \int_{\Omega} \left[qU_3 + \frac{\nu}{(1-\nu)\lambda^2} q\delta_{\alpha\beta} U_{\alpha,\beta} \right] d\Omega$$

$$- \int_{\Gamma} U_{\alpha,\beta} \left\{ \frac{D(1-\nu)}{2}\left(u_\alpha n_\beta + u_\beta n_\alpha + \frac{2\nu}{1-\nu} u_\gamma n_\gamma \delta_{\alpha\beta} \right) + \frac{\nu}{(1-\nu)\lambda^2} q \right\} d\Gamma$$

$$+ \int_{\Omega} \frac{D(1-\nu)}{2}\left(u_\alpha U_{\alpha,\beta\beta} + u_\beta U_{\alpha,\beta\alpha} + \frac{2\nu}{1-\nu} u_\gamma U_{\alpha,\beta\gamma}\,\delta_{\alpha\beta} \right) d\Omega = 0 \qquad (27)$$

Where the bold ($\mathbf{Q}_\alpha$) is the shear stress resultant corresponding to (U). The second integral in equation (27) can be decomposed using integration by parts and using equation (21), as follows:

$$\int_{\Omega} \mathbf{Q}_\alpha \left(u_\alpha + u_{3,\alpha}\right) d\Omega = \int_{\Omega} \mathbf{Q}_\alpha u_\alpha d\Omega + \int_{\Omega} \mathbf{Q}_\alpha u_{3,\alpha}\, d\Omega$$

$$= \int_{\Omega} \mathbf{Q}_\alpha u_\alpha d\Omega + \int_{\Gamma} \mathbf{Q}_\alpha u_3 n_\alpha d\Gamma + \int_{\Omega} u_3 \mathbf{Q}_{\alpha,\alpha}\, d\Omega$$

$$= \int_{\Gamma} P_3 u_3 d\Gamma + \int_{\Omega} \left(\mathbf{Q}_{\alpha,\alpha} u_3 - \mathbf{Q}_\alpha u_\alpha\right) d\Omega \qquad (28)$$

One can now replace the result of equation (28) into equation (27) and perform the following substitutes:

$(u_\beta) \rightarrow (u_\alpha\ \delta_{\alpha\beta})$, $(n_\beta) \rightarrow (n_\alpha\ \delta_{\alpha\beta})$

$(u_\gamma) \rightarrow (u_\alpha\ \delta_{\alpha\gamma})$, and $(n_\gamma) \rightarrow (n_\beta\ \delta_{\gamma\beta})$

Then equation (27) can be rewritten as follows:

$$\int_{\Gamma} U_i p_i d\Gamma - \int_{\Gamma} P_3 u_3 d\Gamma + \int_{\Omega} \left(\mathbf{Q}_{\alpha,\alpha} u_3 - \mathbf{Q}_\alpha u_\alpha\right) d\Omega + \int_{\Omega} \left[qU_3 + \frac{\nu}{(1-\nu)\lambda^2} q\delta_{\alpha\beta} U_{\alpha,\beta} \right] d\Omega$$

$$-\int_{\Gamma} \overbrace{\frac{D(1-\nu)}{2}\left(U_{\alpha},_{\beta} + \underbrace{U_{\alpha},_{\beta}\delta_{\alpha\beta}\delta_{\alpha\beta}}_{U_{\beta},_{\alpha}} + \frac{2\nu}{1-\nu}\underbrace{\delta_{\gamma\alpha}\delta_{\gamma\beta}U_{\alpha},_{\beta}}_{U_{\theta},_{\theta}}\delta_{\alpha\beta}\right)}^{\mathbf{M}_{\alpha\beta}} u_{\alpha} n_{\beta} d\Gamma$$

$$+\int_{\Omega} \overbrace{\frac{D(1-\nu)}{2}\left(U_{\alpha},_{\beta\beta} + \underbrace{\delta_{\alpha\beta}U_{\alpha},_{\beta\alpha}}_{U_{\beta},_{\alpha\beta}} + \frac{2\nu}{1-\nu}\underbrace{\delta_{\gamma\alpha}U_{\alpha},_{\beta\gamma}}_{U_{\gamma},_{\gamma\beta}}\delta_{\alpha\beta}\right)}^{\mathbf{M}_{\alpha\beta},_{\beta}} u_{\alpha} d\Omega = 0 \tag{29}$$

Where the bold ($\mathbf{M}_{\alpha\beta}$) is the bending moment stress resultants corresponding to (U). Assuming that the system of (U & P) has zero body forces, i.e. ($\mathbf{q}$=0) and using equations (20) and (21), the equation above becomes:

$$\int_{\Gamma} U_i p_i d\Gamma + \int_{\Omega} \left(\mathbf{Q}_{\alpha},_{\alpha} u_3 - \mathbf{Q}_{\alpha} u_{\alpha}\right) d\Omega + \int_{\Omega} \left[U_3 + \frac{\nu}{(1-\nu)\lambda^2} U_{\theta},_{\theta} \right] q d\Omega$$

$$-\int_{\Gamma} P_3 u_3 d\Gamma + \int_{\Gamma} P_{\alpha} u_{\alpha} d\Gamma + \int_{\Omega} \mathbf{M}_{\alpha\beta},_{\beta} u_{\alpha} d\Omega = 0 \tag{30}$$

or

$$\int_{\Gamma} \left(U_i p_i - P_i u_i\right) d\Gamma + \int_{\Omega} \left[U_3 + \frac{\nu}{(1-\nu)\lambda^2} U_{\theta},_{\theta} \right] q d\Omega$$

$$-\int_{\Omega} \left(\mathbf{M}_{\alpha\beta},_{\beta} - \mathbf{Q}_{\alpha}\right) u_{\alpha} d\Omega + \int_{\Omega} \left(\mathbf{Q}_{\alpha},_{\alpha}\right) u_3 d\Omega = 0 \tag{31}$$

Equation (31) can be regarded as the generalized Betti reciprocal theorem for this problem. Similarly, for elasticity problems, by choosing the system of (U & P) to represent the fundamental solution case and introducing the two-point concept as well as the direction of the applied unit generalized force, one obtain:

$$\int_{\Gamma} \left(U_{ki} p_i - P_{ki} u_i\right) d\Gamma + \int_{\Omega} \left[U_{k3} + \frac{\nu}{(1-\nu)\lambda^2} U_{k\theta},_{\theta} \right] q d\Omega$$

$$-\int_{\Omega} \left(\mathbf{M}_{k\alpha\beta},_{\beta} - \mathbf{Q}_{k\alpha}\right) u_{\alpha} d\Omega + \int_{\Omega} \left(\mathbf{Q}_{k\alpha},_{\alpha}\right) u_3 d\Omega = 0 \tag{32}$$

and

$$\mathbf{M}_{k\alpha\beta},_{\beta}(\xi, \mathbf{x}) - \mathbf{Q}_{k\alpha}(\xi, \mathbf{x}) = -\delta(\xi, \mathbf{x})\delta_{k\alpha} \tag{33}$$

$$\mathbf{Q}_{k\alpha},_{\alpha}(\xi, \mathbf{x}) = -\delta(\xi, \mathbf{x})\delta_{k3} \tag{34}$$

After making use of the properties of the Dirac delta, the final integral equation can be written as follows:

$$-u_i(\xi) + \int_{\Gamma(\mathbf{x})} [U_{ki}(\xi,\mathbf{x})p_i(\mathbf{x}) - P_{ki}(\xi,\mathbf{x})u_i(\mathbf{x})] d\Gamma(\mathbf{x})$$

$$+ \int_{\Omega(\mathbf{X})} \left[U_{k3}(\xi,\mathbf{X}) + \frac{\nu}{(1-\nu)\lambda^2} U_{k\theta,\theta}(\xi,\mathbf{X}) \right] q(\mathbf{X}) d\Omega(\mathbf{X}) = 0 \quad (35)$$

which is the direct boundary integral equation (or generalized Somigliana identity) for an internal point of the shear-deformable plate in bending.

3 Conclusions

In this tutorial, we have described two examples for the formulation of the direct integral equations. The first example dealt with elasticity equations, whereas the second one applied to shear-deformable plates in bending. It can be seen that the derivation steps are systematic, however it needs some care when using the indicial notation combined with some skill in condensing some of the algebraic expressions. Most of the integral equations are derived using the above procedures. However others need additional decomposition using the integration by parts as presented in simple form in tutorial 2. Such advanced examples will be covered in future tutorials.

In the next tutorial, we will discuss the derivation and the properties of fundamental solutions.

4 Exercise

Derive the direct boundary integral equation for the shear-deformable plate bending resting on the two-parameter Pasternak foundation model. The governing differential equations for this model are given as follows:

$$M_{\alpha\beta,\beta} - Q_\alpha = 0 \quad (36)$$

$$Q_{\alpha,\alpha} + q - K_f u_3 + G_f \nabla^2 u_3 = 0 \quad (37)$$

where (K_f) and (G_f) are two foundation constants.

References and further reading

[1] Brebbia, C.A. & Dominguez, J., Boundary Elements: An Introductory Course, WIT Press, Southampton, UK & McGraw Hill, 1989.

[2] Rashed, Y.F., Boundary Elements Formulations for Thick Plates, WIT Press, Southampton, UK, 2000.