Computer Assisted Mechanics and Engineering Sciences, **4**: 301–304, 1997.

An Appreciation of Erich Trefftz

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Vita

Erich Trefftz was born on February 21, 1888 in Leipzig, Germany. His father Oskar Trefftz was a merchant, and also his mother Eliza, née Runge descended from a merchant family. His grandmother was British, and such he had early contacts to his British relatives with the side effect that he could speak English fluently. In 1990, the family moved to Aachen where he got his *matura* in a humanistic gymnasium in 1906. In the same year he began his studies in the faculty of mechanical engineering of the Technical University of Aachen, but changed to mathematics already after half of a year. It is essential for the whole scientific career of Erich Trefftz that he got access to mathematics through a technical discipline. Later, he became one of the eminent applied mathematicians and mechanicians of Germany with major interest in theoretical and numerical problems of continuum mechanics. Remarkably, not many mathematicians worked in applied topics during the second halves of the 19th century, different from 17th, 18th and the first halves of 19th century, where famous mathematicians like Leibniz, Newton, the Bernoullis, Euler, Lagrange, Cauchy and Gauss were stimulated by physical problems for their mathematical discoverer. With Weierstrass, Dedikind and Cantor, the inner logical development of the structures of mathematics was progressed in Germany. But with the begin of the 20th century a new drive towards applied and numerical mathematics began by Ritz and Galerkin and also by Carl Runge, the uncle of Erich Trefftz. Runge postulated that a mathematical problem can only be said to be solved totally if — at the end — results can be also produced in the form of numbers and this challenge guided Erich Trefftz always.

In 1908 Trefftz moved from Aachen to the University of Göttingen, and that time the Mecca of Mathematics and Physics. Here after Gauss, Dirichlet, Riemann now Hilbert, Klein, Runge and Prandtl created a continuous sequence of world–wide first class mathematical progress. Such Trefftz' most important teachers were Carl Runge, David Hilbert and also Ludwig Prandtl, the genious mechanician in modern fluid– and aerodynamics. Trefftz spent one year at the Columbia University, New York, and then left Göttingen for Strassburg to study under the guidance of the famous Austrian applied mathematician Richard von Mises who founded the GAMM (Gesellschaft für Angewandte Mathematik und Mechanik) in 1922 together with Ludwik Prandtl in Göttingen. Richard von Mises was also the first editor of ZAMM (Zeitschrift für Angewandte Mathematik und Mechanik). Trefftz had live–long friendly relationship to von Mises who, being a Jew, had to leave Germany in 1933. Trefftz felt and showed outgoing helpful solidarity and friendship to von Mises, and he clearly was in expressed distance to the Hitler regime until he early died in 1937. But feeling the responsibility for Science he took over the presidentship of GAMM, and he became the editor of ZAMM in 1933 in full accordance with Richard von Mises.

The academic career began with his doctoral thesis in Strassburg in 1913 where he solved a mathematical problem of hydromechanics. He was a soldier in the first world war, but already in 1919 he got his habilitation and became a full professor of mathematics in Aachen.

In 1918 Erich Trefftz married Friede Offermann, and they got five children. The marriage was very happy, and it was of great importance in his life.

In the year 1922 he got a call as a full professor with a chair in the faculty of mechanical engineering at the Technical University of Dresden. There he became responsible for teaching and research of strength of materials, theory of elasticity, hydrodynamics, aerodynamics and aircraft technique. In the following years he mainly published papers on aero– and hydrodynamics and then devoted his interests more to theory of elasticity, stability theory and structural problems.

In 1927 he moved from the engineering to the mathematical and natural science faculty, being appointed there as a chairholder in Technical (Applied) Mechanics. Already in 1929 Erich Trefftz became an honourable doctor (Dr–Ing.) of the Technical University of Stuttgart. He was an excellent and inspired teacher, joining his deep theoretical and practical understanding of mechanics with mathematical rigour in a broad area of technical problems, but his special inclination always belonged to aerodynamics.

Erich Trefftz died nearly 49 years old on January 21, 1937 in Dresden in consequence of a malicious disease. An expressive bust in the Willers building of the Technical University of Dresden is a reminiscent to a great mechanician of this century.

This part of the appreciation contains extracts of a mathematical thesis in German language written by L.P. Klotz in 1979 at the Technical University of Dresden.

Scientific achievements in applied mathematics and with theory of elasticity including buckling problems

Erich Trefftz had very successful research in hydrodynamics, applied mathematics, vibration theory and theory of elasticity including stability and other subjects with outstanding publications, see also the papers by R. Grammel '*Das wissenschaftliche Werk von Erich Trefftz*' in ZAMM, vol. 18 (1938), No. 1, pp. 1–11, and of C. Biezeno '*Erinnerungen an Erich Trefftz*', in ZAMM, vol. 42 (1962), No. 9, pp. 369–372.

This appreciation is restricted to the fields of applied and numerical mathematics and to the theory of elasticity with related topics, viewing the main contents of the 1996 Cracow conference on 'Trefftz method' with new ideas and developments in the frame of direct (discrete) variational methods with trial– and test–functions in finite subdomains of the boundary and also of the field domain, i.e. Trefftz–type finite boundary element– and field–element–methods and couplings of them.

Trefftz' **mathematical work** was motivated by technical applications. An early paper [4] (his list of publications is given at the end) deals with an improvement of the Picard method as a successive solution approximation of ordinary differential equations and systems of them by introducing the arc length of the integral curve as a new variable. A very witty paper is [7] and concerns the solution of the potential and bipotential equations in a nearly circular domain with given boundary data, using the fundamental solutions of the operators. The method can be applied to elliptic operators with symmetric Green's functions. The main idea is a successive correction of the circular boundary by an intermediate Green's function.

The main paper in the contest of our conference is [21], '*Ein Gegenstück zum Ritzschen Verfahren*' (a counterpart to Ritz' method) lectured 1926 in Zürich on the 2nd International Congress of Technical Mechanics (later called the ICTAM — congress of IUTAM). In [25] we find amendments to [21]. Trefftz treated in [21] St. Venant's theory of torsion (governed by the Poisson equation for the torsional stress function) for prismatic bars with arbitrary full cross–sections. Starting from Dirichlet's minimum principle, a corresponding principle and its discrete form are given for test functions which fulfill the field equations but not the geometrical boundary conditions. Such "Trefftz Method" is a direct variational principle for the approximate fulfillment of the boundary conditions. This method can be seen as a generalization of the developing the particular solution of the inhomogeneous problem in a series for fulfilling the boundary conditions. The minimum property of the energy functional can easily be shown and also the convergence of the discrete problem. Trefftz also relaxed the differentiability requirements of the trial functions in the domain by introducing corresponding Lagrangian multipliers but then, of course, the dimensional reduction of the discrete problem gets at least partly lost.

It should be mentioned that Trefftz' method is not convenient for bipotential operators without enhancement; there one needs squares regularization of the boundary conditions, containing at least one of the two residua at the boundary. Otherwise the consistency of the 2nd derivatives of the approximate solution (the bending moments in the case of the Kirchhoff plate equation) does not hold. This problem was treated in my doctoral thesis (1964) and the habilitation thesis (1969) and at length in the doctoral thesis of P. Weidner (1967).

It is obvious that there is a connection of Trefftz' method with the discrete boundary integral equation method (BEM) but the latter has the advantage of using the kernel functions which provides nice mathematical properties. Also coupling of Trefftz type BEM and FEM (Ritz method using subdomains) and furthermore hybrid and hybrid–mixed finite elements can be traced back to the generalization of Trefftz' method.

E. Trefftz' most extensive work concerns the **theory of elasticity**. He treated the coupled bending and torsional buckling of a prismatic bar in full generality for axial and transverse loading, using an integral equation representation [17]. He also found an elegant version for the torsional problem of a prismatic bar with polygonal cross–section [8, 25], using a conforming mapping of the given polygon onto another one which surrounds the unit circle. The special problem of injumping corners of the cross–section (where the shear stress becomes infinite) was approximated by locally rounded corner lines [15], and in [18] he used Prandtl's soap film analogy to implement a yield limit for plastifying material.

The most important contributions to elasticity are the text book–like sections in [24] (in the 'Handbuch der Physik', Vol. 6) and in [23], and I recommend to read them also today, because they comprehensively present the theory, the variational principles and main solutions. The general integration theory in elastostatics is based on the reciprocal theorems of Betti and Maxwell and on the Somigliani identities, using such main ideas of potential theory and tracing already in principle the BEM. He gives the existence proofs of solutions and includes then discrete variational methods of Ritz and himself without using kernel functions, i.e. without using the actual idea of BEM of today. But one has to realize that the concept of test– and trial–functions on finite subdomains was not yet known at that time such that he could not realize a true discrete BEM.

He always had special interest in stability problems of elasticity theory, and for this he extensively used the variational calculus. He introduced the total potential energy with finite elastic deformations in the precritical state and derived the condition for the critical load parameter by the limit of a state minimum of the energy [27, 28]. Also the derivation of Love's shell equations by applying Dirichlet's principle and demonstrating the strategical use of Lagrangean parameters can be found in [30].

A nice piece of work is [31] where he proves that the coordinates of the shear centre of the cross–section of a prismatic bar fulfill the minimum energy condition of the free torsional axis for St. Venant torsion.

And finally the buckling problem of a rectangular compressed clamped thin plate was solved in [32] by variational calculus. From the upper bound property of Ritz' method he achieved to get a lower bound for the wanted critical load parameters by relaxing the boundary conditions using Fourier–trial functions. He also solved the plate buckling problem caused by shear loading [34] with upper and lower bound.

Summarizing, it can be observed that all his papers tackled new complicated problems of general interest. He showed mastership in the integration of physical and mathematical tools. The discussion of results and special cases did not leave out any detail, and his goals always were clearness and simplicity as far as possible.

And there was a complete harmony of the scientist and the personage Erich Trefftz. His fairness, veracity and courage, his expressed sense of humour and a sincere affection to his family, his friends and his students induced C. Biezeno to say about him 'Erich Trefftz was a man'. It is a great pity that he died so early in the creative period of his life.

P.S.

As a German professor and the chairman of the German Committee of Mechanics (DEKOMECH) it is my privilege and pleasure to thank A.P. Zieliński from the Cracow Technical University for organizing a conference on the memory of Erich Trefftz.

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References

[1] Über die Kontraktion kreisförmiger Flüssigkeitsstrahlen. Diss. Strassburg 1913 (Leipzig 1914), *Z. Math. Phys.*, **64**: 34–61, 1916.
[2] Graphische Konstruktion Joukowischer Tragflächen, *Z. Flugtechn. Motorluftsch*, **4**: 130–131, 1913.
[3] Über Längsstabilität und Längsschwingungen von Flugzeugen, (with Th. von Kármán). *Jahrb. Wiss. Ges. Luftf.*, **3**: 116–138, 1914/15.
[4] Über die Konvergenz des Picardschen Verfahrens der sukzessiven Approximation bei gewöhnlichen Differentialgleichungen. *Math. Ann.*, **76**: 327–332, 1915.
[5] Zur Frage der Holmfestigkeit. *Z. Flugtechn. Motorluftsch.*, **9**: 101–103, 1918.
[6] Potentialströmung um gegebene Tragflächenquerschnitte, (with Th. von Kármán). *Z. Flugtechn. Motorluftsch.*, **9**: 111–116, 1918.
[7] Eine methode zur Lösung der Randwertaufgabe partieller Differentialgleichungen. *Math. Ann.*, **79**: 246–264, 1919.
[8] Über die Torsion prismatischer Stäbe von polygonalen Querschnitt. *Math. Ann.*, **82**: 97–112, 1921.
[9] Zur Prandtlschen Tragflächentheorie. *Math. Ann.*, **82**: 306–319, 1921.
[10] Prandtlsche Tragflächen– und Propeller–Theorie. *ZAMM*, **1**: 206–218, 1921.
[11] Grundwasserströmung in einem abfallenden Gelände mit Abfanggraben, (with L. Hopf). *ZAMM*, **1**: 290–298, 1921.
[12] Über die Schwingungen des Schachtlotes. *Mitt. Markscheidews.*, 1–11, 1921.
[13] Das statische Gravitationsfeld zweier Massenpunkte in der Einsteinschen Theorie. *Math. Ann.*, **86**: 317–326, 1922.
[14] Prandtlsche Tragflächen– und Propellertheorie. Vortr. aus d. Geh. d. Hydro– u. Aerodyn., 34–46. Innsbruck 1922, Berlin 1924.
[15] Über die Wirkung einer Abrundung auf die Torsionsspanungen in der imneren Ecke eines Winkeleisens. *ZAMM*, **2**: 263–267, 1922.
[16] Schwingungsprobleme und Integralgleichungen. *Math. Ann.*, **87**: 307–317, 1922.
[17] Allgemeine Theorie der Knickung des geraden Stabes. *ZAMM*, **3**: 272–275, 1923.
[18] Über die Spannungsverteilung in tordierten Stäben bei teilweiser Überschreitung der Fliessgrenze. *ZAMM*, **5**: 64–73, 1925.
[19] Zur berechnung der Stabilität periodischer Bewegungsvorgänge. *ZAMM*, **5**: 473–475, 1925.
[20] Zu den Grundlagen der Schwingungstheorie. *Math. Ann.*, **95**: 307–312, 1926.
[21] Ein Gegenstück zum Ritzschen Verfahren. Verh. d. 2. Intern. Kongr. f. Techn. Mech., 131–137, Zürich 1926.
[22] Ein Diffusionsproblem. Sitzungsb. Heidelb. Akad. Wiss. math.–phys. K1, 28–38, 1927.
[23] Mathematische Grundlagen der Elastizitätstheorie. Probleme des Elastischen Gleichgewichts, Dynamische Probleme der Elastizitätstheorie. *Riemann–Webers Differentialgleichungen der Physik*, Bd. 2, 598–734, Braunschweig 1927; 2. Aufl., Bd. 2, 240–373, Braunschweig 1935.
[24] Mathematische Elastizitätstheorie. *Handb. d. Physik*, Bd. 6, 47–140, Berlin 1928.
[25] Konvergenz und Fehlerabschätzung beim Ritzschen Verfahren. *Math. Ann.*, **100**: 503–521, 1928.
[26] Zur Berechnung der Schwingungen von Kurbelwellen. Vortr. a. d. Geb. d. Aerodyn. u. verw. Geb., 214–219, Aachen 1929, Berlin 1930.
[27] Über die Ableitung der Stabilitätskriterien des elastischen Gleichgewichtes aus der Elastizitätstheorie endlicher Deformationen, Verh. d. 3. Intern. Kongr. f. Techn. Mech., Bd. 3, 44–50, Stockholm 1930.
[28] Über Fehlerabschätzung bei Berechnung von Eigenwerten. *Math. Ann.*, **108**: 595–604, 1933.
[29] Zur Theorie der Stabilität des elastischen Gleichgewichts. *ZAMM*, **13**: 160–165, 1933.
[30] Ableitung der Schalenbiegungsgleichungen mit dem Castiglianoschen Prinzip. *ZAMM*, 101–108, 1935.
[31] Über den Schubmittelpunkt in einem durch eine Einzellast gebogenen Balken. *ZAMM*, **15**: 220–225, 1935.
[32] Die Bestimmung der Knieklast gedrückter, rechteckiger Platten. *ZAMM*, **15**: 339–344, 1935.
[33] *Graphostatik* (90 pages). Leipzig und Berlin 1936.
[34] E. Trefftz and F.A. Willers. Die bestimmung der Schubbeanspruchung beim Ausbeulen rechteckiger Platten. *ZAMM*, **16**:
336–344, 1936.
[35] Über die Tragfähigkeit eines längsbelasteten Plattenstreifens nach Überschreiten der Beullast, (with K. Marguerre). *ZAMM*, **17**: 85–100, 1937.
[36] Berechnung der Zirkulation für die gerade, tragende Linie. *ZAMM*, **18**: 12–20, 1938.
[37] Ableitung der Verzerrungskomponenten und der Gleichgewichtsbedingungen in Zylinder– und Polarkoordinaten. *ZAMM*, **18**: 91–92, 1938.